



The Math Forum: Problems of the Week

Problem Solving and Communication

Activity Series

Round 18: Change the Representation

All math problems, whether they are word problems, arithmetic problems, equations to solve, etc., come to us in a particular representation. Word problems are represented in story form, using words and often referencing a particular context. Arithmetic problems are represented numerically. Equations are represented using mathematical symbols. Each representation has benefits to the problem solver. For example, word problems allow students to apply their knowledge of the given context, which can allow them to check that their approaches are reasonable. Numeric and symbolic representations can make it easy for students to manipulate objects in the problem, and to quickly see patterns. Visual and physical representations, such as manipulatives, diagrams, and graphs, can often help students gain new insights into the problem and provide them with additional tools for solving it. Changing the representation can mean use of a different form of representation (e.g. using a line drawing for a word problem) or it can mean trying different ways of presenting the information in the same form (e.g. rewriting all of the numbers as fractions with a numerator of 1). Considering multiple representations and choosing representations that fit the problem well are important problem-solving skills.

The activities below help students to brainstorm and work with multiple representations, and compare what they learned about the problem using their different representations.

The activities are written so that you can use them with problems of your choosing. We include a separate section afterward to show what it might look like when students apply these activities to the current PreAlgebra Problem of the Week.

Problem-Solving Goals

Changing the representation of a problem can help problem-solvers:

- Strengthen their understanding of the problem.
- Gain new insights into the problem or solution.
- Provide additional tools for solving the problem.
- Find multiple solution paths, leading to deeper mathematical connections.

Communication Goals

Changing the representation of a problem requires that students change how they communicate about the problem and find different ways to express the same idea or information. They might:

- Paraphrase the problem in terms of a different representation.
- Re-tell the story of the problem with a different context.
- Organize the numerical and calculation strategies using a table or other organizational method.
- Use mathematical symbols to restate the problem succinctly.
- Use diagrams to communicate the math in the problem.
- Represent the problem graphically.

Activities

I. Brainstorm Representations

Format: students working individually or in pairs, then sharing with groups of 4-6.

There are many ways to represent math problems and mathematical ideas. Math problems are often represented in words. Math can be represented visually, through graphs, diagrams, and sketches. Tables and expressions can be used to represent math ideas numerically. Mathematical ideas are often represented symbolically, with operations, numbers, variables, and functions. Each representation can help you understand and solve the problem in different ways. Problems represented in words help you to make sense of the problem and use your knowledge of real-world

situations. Graphical representations can lead to new insights or problem-solving methods. Numerical representations can help you find patterns and generate strategies. Symbolic representations represent math ideas clearly and succinctly, and help you to manipulate the mathematical objects.

Sample Activity

Work individually or in pairs to begin filling in the blanks of the following prompts for just a few minutes. Then share ideas with the larger group of 4-6 students. The first question asks you to think about the math ideas in the problem, which might get you thinking about other representations you know. The second asks you to think as creatively as you can.

1) The main mathematical ideas and relationships in this problem are _____.

2) I could represent this problem by: (drawing a picture, using some blocks, making a graph, writing some equations, telling a different story, writing the numbers or expressions in a different way).

3) My representation might look like _____.

Share your thoughts with your group. If the ideas you hear spark other ideas, record those too. Try to brainstorm as many possible representations as you can.

Suggestions

These suggestions can be used as a place to start with a student who is struggling.

- Think of or make up another story or situation that could be used to present a math problem just like this.
- Think of manipulatives like blocks or chips or algeblocks or fraction bars or the number line or a drawing that you could use to model and explore this problem.
- Think about substitutions you could make for some of the quantities in the problem (rewriting whole numbers as fractions; express all of the quantities in terms of the smallest item; use a different expression that has the same value but might make the calculations easier)

Key Outcomes

- Identify key mathematical ideas in the problem that can be represented.
- Think of ways that mathematical ideas are sometimes represented.
- Generate multiple possible representations for the problem to be solved.

II. Representing

Format: Individually and then in pairs or teams.

The focus in this activity is on using writing to organize the problem solving activity, to notice patterns or ideas that make the solution possible, and to ask specific questions that need to be answered in order to make progress,

Sample Activity

Step 1: Pick the representation that seems most useful or stimulating and play it out in detail. Use your new representation to explore the problem and solve it if you can.

Step 2: Explain to your partner how your new representation works:

- What did you notice about the problem when you changed the representation? What new information, relationships, patterns, and approaches occurred to you?
- What have you been able to figure out for a solution so far?
- If you are stuck, which representation shows best where and why you are stuck? Ask your partner(s) if they have a way of representing the problem that helps you get unstuck.

Step 3: Check your solution with the original problem to make sure it fits with all of the information and constraints of that situation.

Key Outcomes

- Play out a particular representation as fully as possible.
- Consider multiple representations when you get stuck.
- Work with others to see multiple perspectives and fresh ideas.

III. Comparing

Format: Students working individually or pairs and then sharing with the whole group.

Sample Activity:

Step 1: Within your group of 4-6, compare the different solution paths you generated. Did different representations lead to different insights? In what ways are the different solutions similar?

Step 2: Select the representations that provided the most insight into the problem or key steps in the solution. Prepare a presentation for the class on how each representation helped you get to a key insight into the problem.

Step 3 (optional): Submit your write-up to the PoW online.

Step 4 (optional): Use a jigsaw or gallery walk format to share your explanation with classmates and to appreciate their insights.

Key Outcomes:

- Compare insights generated by multiple representations. Identify the similarities and the differences in the contributions from each representation.
- Evaluate how well different representations fit a given problem. Figure out how to recognize when a particular way of changing the presentation of the problem will be useful.

Examples: Suli's Sense (PreAlgPoW)

The goal of these lessons is for the students to reflect on their own process in developing simpler versions of a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "prealgpow-teachers" discussion group (<http://mathforum.org/kb/forum.jspa?forumID=527>) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

If we do facilitate by asking some strategy questions, then at the end of the session we often ask students to notice the questions and suggestions we asked so that they can begin to do that for themselves: Which were helpful? Could you see how you could use these with other problems? Which questions would you put on a class list of "Ways to get Unstuck in Changing the Representation"?

I. Brainstorm Representations

- The mathematical ideas in this problem are:
 - Dealing with data.
 - The mean of a set of data.
 - The difference between each actual high temperature and the mean.
 - The sum of those differences equals 0.
- I could represent this problem by:
 - Acting it out with tiles.
 - Acting it out with numbers.
 - Writing equations.
- My representation might look like:
 - Groups of tiles that have a mean of 72.
 - A set of temperatures that have a mean of 72° F.
 - An equation that relates a set of temperatures to their mean.

II. Representing

- **Student 1:** I represented the problem with tiles:

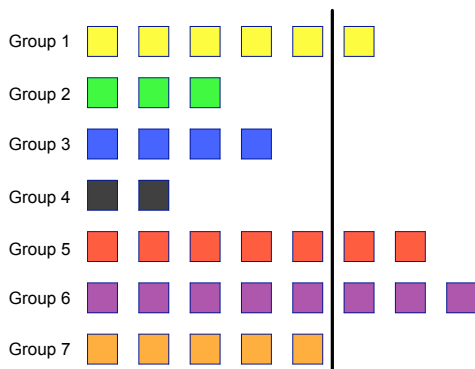
The problem is about means. I decided to investigate means by making different-sized groups of tiles and exploring the differences between the number of tiles in each group and what the mean (average) number of tiles in the groups is.

A mean of 72 is a lot of tiles. I decided to divide 35 tiles among 7 groups (like the 7 days in Suli's problem).



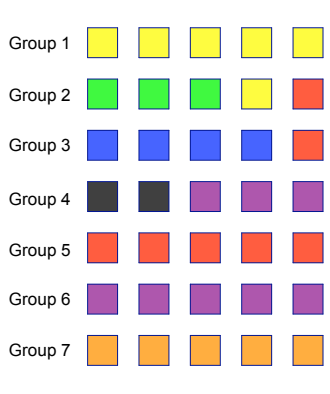
I calculated the mean number of tiles in each group:

$$(6 + 3 + 4 + 2 + 7 + 8 + 5)/7 = 35/7 = 5.$$



Then I decided to look at the differences between each group and the mean. Group 1 (yellow) had 1 more tile than the mean. Group 2 (green) has 2 fewer tiles. Group 3 (blue) has 1 fewer tile. Group 4 (black) has 3 fewer tiles. Group 5 (red) has 2 more tiles. Group 6 (purple) has 3 fewer tiles. Group 7 (orange) has exactly the mean number of tiles.

I decided to rearrange the tiles in the groups so each group had the mean number of tiles (5). I moved tiles from the groups that had more tiles than the mean to the groups that had tiles less than the mean:



I took away 1 tile from Group 1 (yellow).

I added 2 tiles to Group 2 (green).

I added 1 tile to Group 3 (blue).

I added 3 tiles to Group 4 (black).

I took away 2 tiles from Group 5 (red).

I took away 3 tiles from Group 6 (purple).

I was able to make each group into a new group with exactly the mean number of tiles. It makes sense to me that 5 is the mean number of tiles and also the number that would be in each group if the tiles were divided evenly.

I had exactly the right number of tiles to give to each group that had less than 5 tiles. I had 6 tiles from groups with more than 5 tiles (1 from Group 1 (yellow) + 2 from Group 5 (red) + 2 from Group 6 (purple)) and exactly 6 tiles were needed by the other groups (Group 2 (green) needed 2, Group 3 (blue) needed 1, and Group 4 (black) needed 3).

Comparing this to Suli's situation, the difference between each high temperature and the mean is represented by the number of tiles I needed to add or take away from a group so that its tiles will be equal to the mean.

I think that the differences between the mean and the temperatures higher than the mean will always equal the differences between the mean and the temperatures lower than the mean. These differences cancel each other out.

Student 2: I am going to represent the problem using a set of data:

I know that the mean high temperature for the week was 72°F . I don't know the high temperature for each day. I can guess the temperatures and check to see if their mean is 72°F :

Day	Temperature $^{\circ}\text{F}$
Monday	80
Tuesday	30
Wednesday	40
Thursday	50
Friday	60
Saturday	70
Sunday	90
Mean	60

I got a mean of 60°F , which is lower than 72°F by 12°F . I think I can make the mean higher by raising one of the temperatures. I will make the temperature on Tuesday higher. I will try adding 12°F .

Day	Temperature $^{\circ}\text{F}$
Monday	80
Tuesday	$30 + 12 = 42$
Wednesday	40
Thursday	50
Friday	60
Saturday	70
Sunday	90
Mean	61.7142857

That resulted in a very small increase in the mean. Hmm... I know that when I calculate the mean, I add all the temperatures and divide them by 7. If I want to increase the mean by 12 to become 72, should I add $12 * 7 = 84$ to the temperature on Tuesday? I will try that and see what happens.

Day	Temperature °F
Monday	80
Tuesday	$30 + 84 = 114$
Wednesday	40
Thursday	50
Friday	60
Saturday	70
Sunday	90
Mean	72

Great! I found a set of temperatures that have a mean of 72 °F. I can now do what Suli did and find the difference between each temperature and the mean and see if the differences add up to 0.

Day	Temperature °F	Difference from the Mean °F
Monday	80	8
Tuesday	114	42
Wednesday	40	32
Thursday	50	22
Friday	60	12
Saturday	70	2
Sunday	90	18
	Total	138

Hmm! The differences don't add up to 0. Could it be that Suli's result was a coincidence? But what kind of data set would give her zero? These differences can never add up to zero. The only way the differences can add up to 0 is if some of them are positive and some are negative. Hey, maybe Suli considered the difference negative when the temperature was less than the mean and positive when the temperature was higher than the mean. I can try that and see what happens:

Day	Temperature °F	Difference from the Mean
Monday	80	8
Tuesday	114	42
Wednesday	40	-32
Thursday	50	-22
Friday	60	-12
Saturday	70	-2
Sunday	90	18
	Total	0

The sum of the differences is 0. I think that Suli's result wasn't a coincidence and that if a set of values has a mean of 72 then the sum of differences between each value and 72 will be 0.

This makes sense because when we calculate the mean, we add up all the values and divide them by the number of values. It is as if we mix them all together and even them out by adding to the low values and taking away from the high values. Whatever we add up has to be equal to what we take away.

- **Student 3:** I am going to represent the problem using equations:

It is hard for me to write an equation for the mean of the temperatures over the 7 days of the week. If I wrote an equation for the temperatures over two days, I can probably understand more about what happens over the whole week. I will represent the first temperature as x and the second temperature as y . The mean of these two temperatures is:

$$\text{mean} = \frac{x+y}{2}$$

The difference between x and the mean is:

$$x - \frac{x+y}{2}$$

The difference between y and the mean is:

$$y - \frac{x+y}{2}$$

If I add the differences I will get:

$$x - \frac{x+y}{2} + y - \frac{x+y}{2}$$

Hmm... How does that come out to 0?

I see that $\frac{x+y}{2}$ is being subtracted twice. I think that's the same as subtracting $(x+y)$ once. So I have $(x+y) - (x+y)$.

I think that we will get the same result if we had a set of more than two temperatures.

With three numbers, adding up the differences would look like:

$$x - \frac{x+y+z}{3} + y - \frac{x+y+z}{3} + z - \frac{x+y+z}{3}$$

You subtract $\frac{x+y+z}{3}$ three times, which is the same as subtracting $(x+y+z)$. So you have $(x+y+z) - (x+y+z)$ and you get 0.

You'll always end up subtracting the sum of the numbers from the sum of the numbers and get 0.

Suli's result was not a coincidence. She should always get 0 when she adds the differences between each high temperature and the mean high temperature.

III. Comparing

We compared our solutions and we found that we all looked for ways to represent the mean of a set of values. Although we took three different paths, we all decided that Suli's result was not a coincidence and that she should have got 0 no matter what.

Student 1 and **Student 2** both acted out what Suli did by finding a set of values and finding the sum of the differences between each value and the mean. **Student 1** decided not to work with the exact values in the problem, but to use smaller numbers to help her understand the relationship between the values in the set and their mean.

Student 2 acted out the problem by finding a set of temperatures for the week that have a mean of 72. He started by making a guess for the temperatures and finding their mean. He found the difference between this mean and 72. He was convinced that Suli's result wasn't a coincidence because he also got 0 with a data set he made up, that was probably different from Suli's actual data. He also thought about why it might make sense for the differences to be 0.

Negative and positive numbers were also important in this problem. **Student 2** considered negative and positive numbers for the differences between the temperatures and the mean. **Student 1** represented positive numbers by the tiles she added to the groups and negative numbers by the tiles she took away from them, as she evened out the number of tiles in the groups.

Student 3 represented the problem by an equation for the mean and an equation for the sum of the differences between the temperatures and the mean. Instead of looking at “evening out” the data, she noticed how when you add up the differences from the mean, you are subtracting the total sum of the data from the total sum of the data, so you always get 0.

Both **Student 1** and **Student 3** chose to simplify their representation of the problem: **Student 1** by working with smaller groups of tiles and **Student 3** by making an equation for temperatures over 2 days and then 3 days instead of the temperatures over the whole week.

So far, we all had to say, “we think this will always work in general,” we haven’t proved 100% that the sum of the differences from the mean is always 0, but we are very convinced.