The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Round 17: Playing

When students do the *Noticing/Wondering* activity, we often have them try to group their noticings into “quantities” and “relationships.” With a little practice, students get adept at finding the quantities and the relationships that are explicitly stated in the problem. However, interesting math problems usually have deeper layers of relationships that only emerge as problem solvers “play” with the relationships and quantities.

In the recent activities focusing on *Planning* and *Getting Unstuck*, we began to highlight some of the phases of problem solving, and to show how many of the activities in this series can be used to explore relationships as you begin problem solving or if you get stuck along the way.

Continuing in this vein, *Round 17* focuses on some of the ways problem solvers play with relationships and explore patterns before they delve deeply into a single problem-solving strategy. In order to make clear different aspects of problem solving, we’ve broken the “play” process out somewhat artificially – expert problem solvers move back and forth fluidly between understanding the problem, playing with relationships, and carrying out strategies. However, for purposes of illustration, we think it will be useful to focus on those phases separately.

The activities are written so that you can use them with problems of your choosing. There is no sequence to the activties. Select one or more that seem appropriate or adaptable to your classroom. We include a separate section after the activity descriptions to provide examples of what it might look like when students apply these activities to the current Math PreAlgebra Problem of the Week.

Problem-Solving Goals

Good problem solvers:

- Play and explore as they solve problems.
- Look for deeper and hidden relationships.
- Try to uncover more and more interesting math.
- Try multiple approaches or ways of looking at a single problem.

Communication Goals

Problem solvers use communication as they play to help them:

- Keep track of interesting things they noticed and wondered.
- Represent the problem in new ways.
- Paraphrase the problem.
- Share their own perspectives and ideas and learn from others.

Activities

I. Calculating (and Noticing) as you Go

Sample Activity: Calculate as you Go

Format: Students working in pairs.

When you see quantities in the problem, you may not see how to solve the problem, but you might think of some calculations you could do. Try doing some of these calculations that come into your head, even if you don’t know that they will help you solve the problem.

Be sure to tell your partner:

What you did (what quantities and operations you used).

What the units of the results are (what you are counting or measuring).

As you calculate, notice if you get any interesting results, or if any of the calculations seem particularly helpful.

II. Playing with Strategies

Sample Activity: Speed Dating

Format: students working in groups of three to five.

Materials: strips of paper, pens or pencils, loose-leaf paper.

In order to get the juices flowing and begin to investigate and unearth more relationships, it can be helpful to try a lot of different ideas quickly. See what you notice, but don't get too bogged down in one idea.

1) Each person writes a strategy or short description of something to try on a **strip of paper**. The activity will be more fun if each person chooses a different strategy. Some good examples: *Guess and Check, Change the Representation, Tables and Patterns,* and *Solve a Simpler Problem*.
2) When everyone is ready, each member of your group should pass his/her **strip of paper** to the left. You have three minutes to do what you can with the strategy or idea that you received. Write your work and what you notice and wonder on your own sheet of loose-leaf paper (at the end, you will have ideas from a few different strategies that you can look back at as you work on the problem).
3) After three minutes, stop wherever you are and draw a line or a box around your work, and write the name of the strategy used.
4) Pass the **strip of paper** with the name of the strategy you were working on to your left, and receive a new one on your right. You have three minutes to work on the new strategy. Start a new section on your piece of loose-leaf paper.
5) Repeat Step 4 until you receive the **strip of paper** that you originally wrote the name of your strategy on. Finish by working on that strategy for three minutes.
6) As a group, add any new relationships, patterns, quantities, interesting ideas, or things you are wondering about to your list of noticings and wonderings.

Key Outcomes

- Get a better understanding of the problem by playing with a variety of ideas before solving the problem.
- Identify and pull together the most promising solution path from multiple representations or multiple strategies.
- Discover and note deeper, hidden relationships that emerge as you play with various possibilities.

III. Playing with Clues

Sample Activity: What If…

Format: students working in groups of three to five.

"Clues" is a useful shorthand for the longer phrase "quantities and relationships you noticed or wondered about."

One way to understand how a particular clue is a useful part of the problem is by changing it and noticing how the problem changes. I might change the value of a clue, or even pretend I don't know it. My goal is to focus on what changes in the problem, and how I can use that information to understand the clue better.

1) With your group, go through each clue or set of clues. Write down ways you could change the value of quantities in that clue or the constraint that it imposes.
2) For each clue, play out the problem a little bit with the new value(s). How would the problem change? Would it be easier or harder? What would be easier or harder about it? Would the results be different?
3) Now go through the clues again, this time ignoring one clue or set of clues at a time, pretending that information was never given.
4) How does the problem change when those clues are ignored? Does it make a simpler version you can use to learn more? Does it change the number of possible answers? What else changes?
5) After you've gone through all the clues, look back at the original problem. What new understanding have you gained? Do you see more uses for any of the clues? Do any of the clues seem more necessary (or unnecessary)?

Key Outcomes

- Explore how the problem was constructed.
- Generate additional information and perspective by changing or ignoring clues.
- Gain better understanding of the problem by thinking about simpler (and harder) versions.

IV. Playing with Pictures (and Representations)

Sample Activity: How Else Can We Say It?

Format: students working in groups of three to five (depending on the number of clues in the problem).

Sometimes the clues are said one way, but if you said them (or wrote them or drew them) just a little differently, you would see different relationships. In this activity, try to express the clues as many different ways as you can.

1) Each person picks one clue from the problem and rewrites it or draws a picture of it or somehow changes *how* it's said without changing *what* is said.
2) After a few minutes, each person passes the clue to their left. Try to add another way to say (or draw or represent) the clue, adding to those already written down.
3) Keep passing clues to the left as long as you can come up with new ways to express them.
4) Once you've run out of ideas working individually, hold a group discussion:
    a. Check if any of the different expressions changed *what* the clues really meant. In this activity you don't want to change what the clues mean. You just want to get a new perspective on what they mean.
    b. Did any new information or perspective emerge that helps you see an approach to solving the problem?

Key Outcomes

- Understand the problem better using multiple representations of key information.
- See the problem from a fresh perspective.

Examples: Bartering for Bananas (PreAlgPoW)

The goal of these lessons is for the students to reflect on their own process in exploring the information given in a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and instead focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "prealgpow-teachers" discussion group (http://mathforum.org/kb/forum.jspa?forumID=527) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

I. Calculating (and Noticing) as you Go

Partnership 1

Student 1: Here are some relationships I noticed:

He can get 2 loaves of bread for 5 fish.

He can get 6 oranges for 2 melons.

He can get 1 banana + 3 oranges for 1 loaf of bread.

He can get 14 oranges for 4 loaves of bread.

Student 2: So he could trade his 5 fish for 2 loaves of bread.

Student 1: Then he could trade his two loaves of bread for 7 oranges.

Student 2: Yeah, or for 2 bananas plus 6 oranges.

Student 1: Then he could trade those 6 oranges for 2 melons.

Student 2: Those are all the trades I can see without doing some more noticing and wondering.

Partnership 2

Student 3: Here are some relationships I noticed:

5 fish = 2 loaves of bread

6 oranges = 2 melons

1 loaf of bread = 1 banana + 3 oranges

4 loaves of bread = 14 oranges

Student 4: How about if we try to find what one fish is worth?

Student 3: If 5 fish are equal to 2 loves of bread, we have to find how many times 5 goes into 2.

Student 4: So one fish is equal to 2/5 of a loaf.

Student 3: I think we should try to solve the problem with whole numbers. I don't think Byron will trade 2/5 of a loaf of bread.

Student 4: I agree. But how can we do that?

Student 3: Let's look at the relationships again and see what calculations we can do while keeping the numbers whole.

Student 4: I can see that 4 loaves of bread = 14 oranges. This means that 2 loaves of bread = 7 oranges. Since 5 fish = 2 loaves of bread, then 5 fish = 7 oranges.

Student 3: 1 loaf of bread or 1 fish would equal a fractional number of oranges, so we can't keep working with those calculations.

Student 4: But we can change what we know about 1 loaf of bread into 2 loaves of bread. We know that 1 loaf of bread = 1 banana + 3 oranges so 2 loaves of bread = 2 bananas plus 6 oranges.

Student 4: We know how much bread Byron can get for his fish, and how many oranges, but we don't know anything about trading bread directly for bananas or oranges. I think we are going to need to do something else to figure out how much bananas are worth.

II. Playing with Strategies

Possible Strategies

- Guess and Check.
- Act it Out with fish, bread and fruits.
- Change the Representation.

Results Round 1

- **Guess and Check (3 minutes):**

  Guess the number of bananas Byron will get for his 5 fish.

  Guess 6 bananas.

  What calculations can I do to check our guess?

  Well, 5 fish are worth 2 loaves of bread, so that would mean 6 bananas are worth 2 loaves of bread.

  Then 1 loaf of bread is worth 3 bananas.

  I already know 1 loaf of bread is worth 1 banana plus 3 oranges.

  So that means 3 bananas are worth 1 banana plus 3 oranges, or 2 bananas are worth 3 oranges.

  Can I check that 2 bananas being worth 3 oranges makes sense, using other clues in the problem?

  Well, 5 fish are worth 2 loaves of bread, which are worth 7 oranges. So if 5 fish are worth 6 bananas, then 6 bananas are worth 7 oranges.

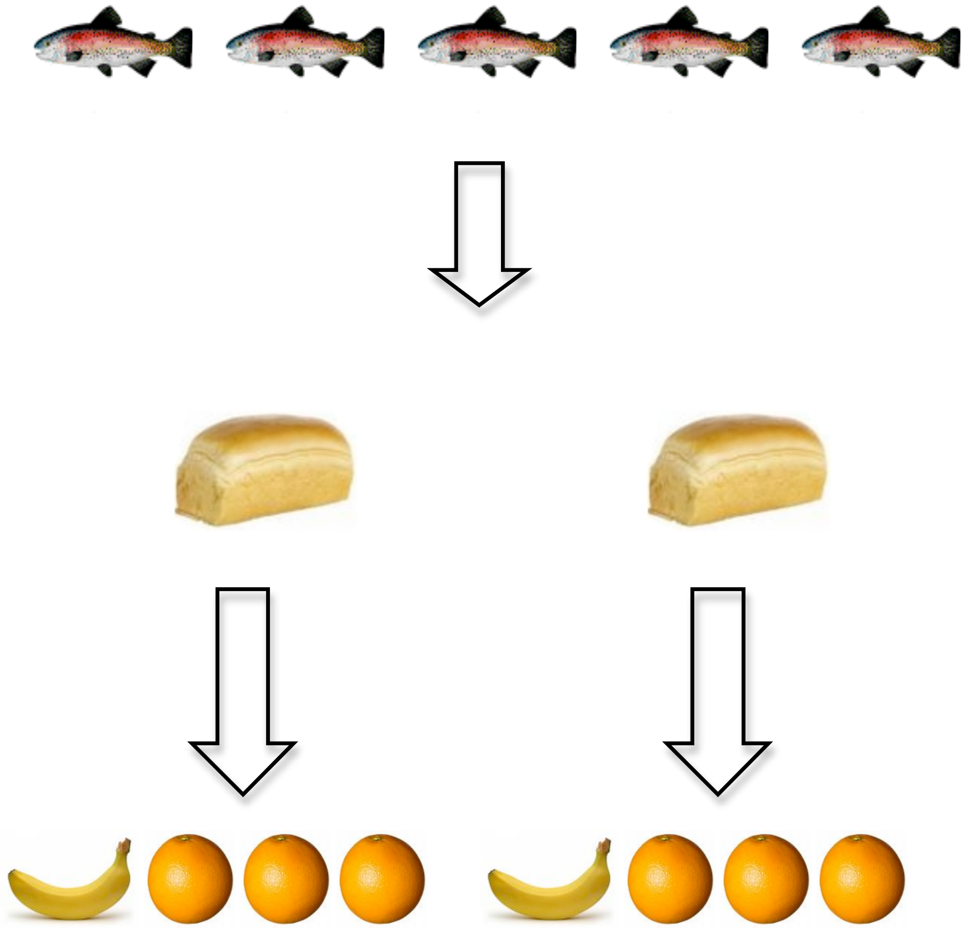
  So it doesn't make sense for 2 bananas to be worth 3 oranges, but 6 bananas to be worth 7 oranges.

  I wonder how I could adjust my guess.

TIME'S UP!

- **Act it Out with fish, fruits and loaves of bread (3 minutes):**

I am ready to act out the problem. I have 5 fishes and I am going to trade them for two loaves of bread:



I can then trade each loaf of bread for a banana plus 3 oranges.

Hmm! What trade should I do next? I know that 4 loaves of bread are worth 14 oranges. Should I trade the oranges to get bread? But I only have 6 oranges.

TIME'S UP!

- **Change the representation (3 minutes):**

I think that I can represent a fish by Letter F, a banana by B, a loaf of bread by L, an orange by O and a melon by M. Using the letters, I can use equations to write how much each of the items is worth:

$5F = 2L$

$6O = 2M$

$1L = 1B + 3O$

$4L = 14O$

Then I can do stuff like add the same thing to both sides or multiply both sides by the same thing to see if I can get 5 F = ? B.

$5F = 2(B + 3O)$ (replace L with 1B + 3O).

$5F = 2B + 6O$.

$5F = 2B + 2M$.

$2.5F = B + M$.

TIME'S UP!

New Noticings and Wonderings

- From *Guess and Check* we noticed that it was somewhat tricky to check our guess. Having a guess for how many bananas was equal to 5 fish helped us write more relationships, but we got different comparisons for oranges and bananas depending on which clues we used.
- The comparisons kept being in terms of oranges and bananas. We wondered if that was an important relationship. We wondered if we could figure out how many oranges were equal to how many bananas without guessing.
- From *Act it Out* we noticed that two trade-ins got us a mix of bananas and oranges. So from acting out the problem we also figured out that the bananas-oranges relationship is really important.
- Using *Act it Out* was hard to get past having a mix of bananas and oranges. There are no trades described that have just bananas in them. We will have to think harder about how to figure out what a banana is "worth."
- From *Change the Representation* we noticed that we could turn all the four clues into four equations with four variables. We might be able to use algebra to solve the equations and find the value of B, but we will have to do a lot of algebra!

III. Playing with Clues

Clues

The Clues we found in the problem were:

- Clue 1: Five fish are worth 2 loaves of bread.
- Clue 2: Six oranges are worth 2 melons.
- Clue 3: One loaf of bread is worth 1 banana plus 3 oranges.
- Clue 4: Four loaves of bread are worth 14 oranges.

Changing Values

1. We decided to change the clue that seemed most complicated, Clue 3. We decided to change it so that we would know the worth of a loaf of bread in bananas only:

- One loaf of bread is worth 1 banana.

We can easily then find that Byron will get 2 bananas for the five fish. It was fun to create an easier problem. We decided to see if we could make the problem harder.

2. So far, we were able to easily ignore Clue 2, since it's the only one with melons in it. We decided to change the clues so we couldn't ignore Clue 2.
If we changed Clue 3 to:

- One loaf of bread is worth 1 banana plus 3 oranges plus 4 melons.

We have then to consider Clue 2. We didn't make the problem too hard because we can easily figure out that 4 melons are worth 12 oranges. We can then use that in Clue 3:

- One loaf of bread is worth 1 banana plus 3 oranges plus 12 oranges.

Noticing and Wondering:

- Changing Clue 3 really made the problem a lot easier. Knowing the value of a loaf of bread in oranges and bananas must be part of what's tricky about the problem. If we could figure out another way to write that clue without changing it, maybe the rest of the problem would be easier.
- When we changed Clue 3 to put melons into it, it was easy to take the melons back out because we could switch them to oranges. Maybe we could change the real clue three by swapping one kind of food for another one in the problem that we know about, like swapping oranges for bread.
- Changing clues to make fractional relationships between some of the foods didn't necessarily mean that you had to cut food in half to do the final trade. Maybe it would be helpful to think about things like 1 fish = 2/5 of a loaf of bread, even though you wouldn't really trade for 2/5 of a loaf.

Ignoring Clues

We think we could probably ignore Clue 2 without changing the problem, since all it does is tell you about how much a melon is worth, and Byron doesn't want melons, and you don't know anything else about them except how many oranges they are worth.

We definitely can't ignore Clue 1, since it's the only clue with fish in it, and that's what Byron has.

We decided to see what would happen if we ignored Clue 4.

- Clue 1: Five fish are worth 2 loaves of bread.
- Clue 2: Six oranges are worth 2 melons.
- Clue 3: One loaf of bread is worth 1 banana plus 3 oranges.

Then five fish are worth 2 bananas + 6 oranges (2 loaves of bread). And those 6 oranges are worth 2 melons. But how can we get rid of the 6 oranges or 2 melons?

6 oranges are worth 2 bananas less than 2 loaves of bread, since 6 oranges plus 2 bananas = 2 loaves of bread.

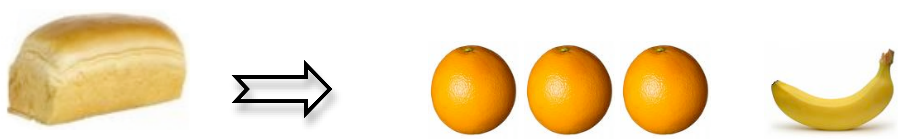
So 5 fish are worth 2 bananas + 2 loaves of bread – 2 bananas, but that just gets us back where we started.

So it must be that knowing Clue 4 (how many oranges 1 loaf of bread is worth) is really important to getting rid of those oranges Byron has after trading bread for oranges and bananas…

IV. Playing with Pictures (and Representations)

Note: only one clue is illustrated here, for purposes of brevity.

Clue: "One loaf of bread is worth the same as one banana plus three oranges."

- Byron can trade one loaf of bread for one banana plus 3 oranges.
- One loaf of bread = 1 banana + 3 oranges.
- One banana is worth 3 oranges less than one loaf of bread.
- Three oranges are worth one banana less than one loaf of bread.
- 
- L = B + 3 O.