Problem Solving and Communication Activity Series

Round 17: Playing

When students do the *Noticing/Wondering* activity, we often have them try to group their noticings into "quantities" and "relationships." With a little practice, students get adept at finding the quantities and the relationships that are explicitly stated in the problem. However, interesting math problems usually have deeper layers of relationships that only emerge as problem solvers "play" with the relationships and quantities.

In the recent activities focusing on *Planning* and *Getting Unstuck*, we began to highlight some of the phases of problem solving, and to show how many of the activities in this series can be used to explore relationships as you begin problem solving or if you get stuck along the way.

Continuing in this vein, *Round 17* focuses on some of the ways problem solvers play with relationships and explore patterns before they delve deeply into a single problem-solving strategy. In order to make clear different aspects of problem solving, we've broken the "play" process out somewhat artificially – expert problem solvers move back and forth fluidly between understanding the problem, playing with relationships, and carrying out strategies. However, for purposes of illustration, we think it will be useful to focus on those phases separately.

The activities are written so that you can use them with problems of your choosing. There is no sequence to the activities. Select one or more that seem appropriate or adaptable to your classroom. We include a separate section after the activity descriptions to provide examples of what it might look like when students apply these activities to the current Geometry Problem of the Week.

Problem-Solving Goals

Good problem solvers:

- Play and explore as they solve problems.
- Look for deeper and hidden relationships.
- Try to uncover more and more interesting math.
- Try multiple approaches or ways of looking at a single problem.

Communication Goals

Problem solvers use communication as they play to help them:

- Keep track of interesting things they noticed and wondered.
- Represent the problem in new ways.
- Paraphrase the problem.
- Share their own perspectives and ideas and learn from others.

Activities

I. Calculating (and Noticing) as you Go

Sample Activity: Calculate as you Go

Format: Students working in pairs.

When you see quantities in the problem, you may not see how to solve the problem, but you might think of some calculations you could do. Try doing some of these calculations that come into your head, even if you don't know that they will help you solve the problem.

Be sure to tell your partner:

> What you did (what quantities and operations you used).
>
> What the units of the results are (what you are counting or measuring).

As you calculate, notice if you get any interesting results, or if any of the calculations seem particularly helpful.

II. Playing with Strategies

Sample Activity: Speed Dating

Format: students working in groups of three to five.

Materials: strips of paper, pens or pencils, loose-leaf paper.

In order to get the juices flowing and begin to investigate and unearth more relationships, it can be helpful to try a lot of different ideas quickly. See what you notice, but don't get too bogged down in one idea.

1) Each person writes a strategy or short description of something to try on a strip of paper. The activity will be more fun if each person chooses a different strategy. Some good examples: *Guess and Check, Change the Representation, Tables and Patterns,* and *Solve a Simpler Problem*.
2) When everyone is ready, each member of your group should pass his/her strip of paper to the left. You have three minutes to do what you can with the strategy or idea that you received. Write your work and what you notice and wonder on your own sheet of loose-leaf paper (this way, at the end, you will have ideas from a few different strategies that you can look back at as you work on the problem).
3) After three minutes, stop wherever you are and draw a line or a box around your work, and write the name of the strategy used.
4) Pass the strategy strip you were working on to your left, and receive a new one on your right. You have three minutes to work on the new strategy. Keep working on your own paper with a new section for each strategy that you do.
5) Pass papers every three minutes until you receive the strategy strip that you originally wrote. Finish by working on that strategy for three minutes.
6) As a group, add any new relationships, patterns, quantities, interesting ideas, or things you are wondering about to your list of noticings and wonderings.

Key Outcomes

- Get a better understanding of the problem by playing with a variety of ideas before solving the problem.
- Identify and pull together the most promising solution path from multiple representations or multiple strategies.
- Discover and note deeper, hidden relationships that emerge as you play with various possibilities.

III. Playing with Clues

Sample Activity: What If…

Format: students working in groups of three to five.

"Clues" is a useful shorthand for the longer phrase "quantities and relationships you noticed or wondered about."

One way to understand how a particular clue is a useful part of the problem is by changing it and noticing how the problem changes. I might change the value of a clue, or even pretend I don't know it. My goal is to focus on what changes in the problem, and how I can use that information to understand the clue better.

1) With your group, go through each clue or set of clues. Write down ways you could change the value of quantities in that clue or the constraint that it imposes.
2) For each clue, play out the problem a little bit with the new value(s). How would the problem change? Would it be easier or harder? What would be easier or harder about it? Would the results be different?
3) Now go through the clues again, this time ignoring one clue or set of clues at a time, pretending that information was never given.
4) How does the problem change when those clues are ignored? Does it make a simpler version you can use to learn more? Does it change the number of possible answers? What else changes?
5) After you've gone through all the clues, look back at the original problem. What new understanding have you gained? Do you see more uses for any of the clues? Do any of the clues seem more necessary (or unnecessary)?

Key Outcomes

- Explore how the problem was constructed.
- Generate additional information and perspective by changing or ignoring clues.
- Gain better understanding of the problem by thinking about simpler (and harder) versions.

IV. Playing with Pictures (and Representations)

Sample Activity: How Else Can We Say It?

Format: students working in groups of three to five (depending on the number of clues in the problem).

Sometimes the clues or the problem are said one way, but if you said them (or wrote them or drew them) just a little differently, you would see different relationships. In this activity, try to express the clues or the problem itself as many different ways as you can.

1) Each person picks one clue from the problem or the problem statement and rewrites it or draws a picture of it or somehow changes *how* it's said without changing *what* is said.
2) After a few minutes, each person passes the clue to their left. Try to add another way to say (or draw or represent) the clue, adding to those already written down.
3) Keep passing clues to the left as long as you can come up with new ways to express them.
4) Once you've run out of ideas working individually, hold a group discussion:
    a. Check if any of the different expressions changed *what* the clues really meant. In this activity you don't want to change what the clues means. You just want to get a new perspective on what they mean.
    b. Did any new information or perspective emerge that helps you see an approach to solving the problem?

Key Outcomes

- Understand the problem better using multiple representations of key information.
- See the problem from a fresh perspective.

Examples: Regional Ratios (GeoPoW)

The goal of these lessons is for the students to reflect on their own process in exploring the information given in a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and instead focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "geopow-teachers" discussion group (http://mathforum.org/kb/forum.jspa?forumID=529) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

I. Calculating (and Noticing) as you Go

Partnership 1

Student 1: I can think of a perimeter calculation: side + side + side = side + side + side + side + side + side.

Student 2: How could those be equal?

Student 1: The problem says the triangle and the hexagon have the same perimeter, so if you add up all the triangle sides you should get the hexagon sides.

Student 2: That means the triangle sides must be twice as long as the triangle sides.

Student 2: How do you know?

Student 1: From your calculation: 3 triangle sides = 6 hexagon sides. So using ratios I figured 1 triangle side = 2 hexagon sides. A triangle side is twice as long as a hexagon side.

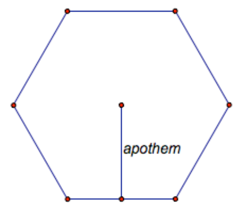
Partnership 2

Student 3: The area of a triangle is (base * height)/2

Student 4: The area of a regular hexagon is (perimeter * apothem)/2

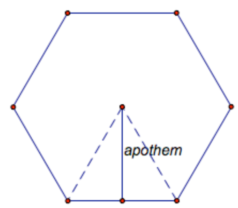
Student 3: What's an apothem?

Student 4: It's sort of like the radius. It's the distance from the center of the hexagon to the middle of one of the sides. Like this:



Student 3: How do you find out how long it is?

Student 4: It's part of this equilateral triangle. So you can use all those 30-60-90 triangle and equilateral triangle tricks on it.



II. *Playing with Strategies*

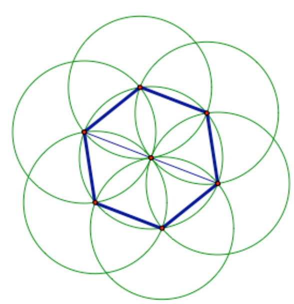
Possible Strategies

- Draw a Picture
- Make a Mathematical Model
- Guess and Check

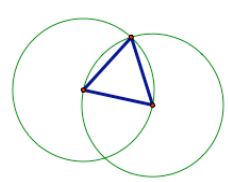
Results

- **Draw a Picture:**

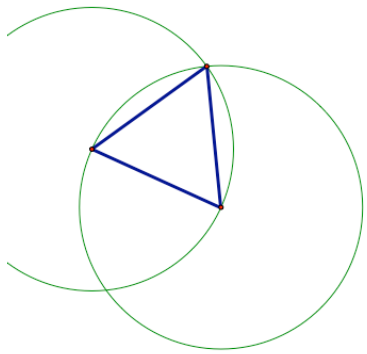
I started by using a compass to draw an accurate hexagon:



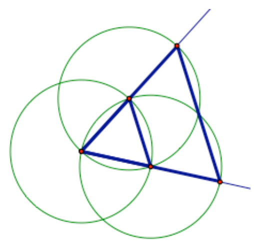
Then I tried to use the compass to draw a triangle with the same perimeter:



That triangle definitely only had half the perimeter of the hexagon, because its sides were the same length as the hexagon's (I kept my compass at the same setting), but there are only three of them. So I set my compass twice as wide by setting it to the diameter of central circle in the hexagon:



I wondered how I could compare my big triangle with the smaller triangle I had made originally. I thought about how if I just doubled the length of each side of the original small triangle, I could do that. So I used my compass to draw more circles to help me see the new, extended triangle.



TIME'S UP!

- **Make a Mathematical Model:**

Quantities:

Perimeter (both equal)

Side lengths

of the triangle (3, all equal)

of the hexagon (6, all equal)

Area of the hexagon

Area of the triangle

Ratio of the areas

Relationships:

3(triangle side length) = Perimeter

6(hexagon side length) = Perimeter

3(triangle side length) = 6(hexagon side length)

triangle side length = 2(hexagon side length)

Area of the triangle = $\frac{base * height}{2}$

TIME'S UP!

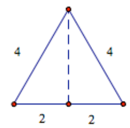
- **Guess and Check:**

This is a hard one – let's see what I can do… I don't know the ratio of the areas, or the areas, or the side lengths, or even what the perimeter is. I could guess a perimeter and see what the side lengths and areas are.

What if the perimeter is 12?

Then each side of the hexagon is 2, and each side of the triangle is 4.

If each side of the triangle is 4, the area is $\frac{base * height}{2}$. I need to find the height.



The height makes a 30-60-90 right triangle that I can use.

TIME'S UP!

New Noticings and Wonderings

- When I *Drew a Picture* I noticed that when I drew a triangle whose sides were equal to the hexagon's sides, it looked like it was 1/6 of the hexagon. I wonder if the triangle with the same perimeter, whose sides are twice as big, is 1/3 of the hexagon?
- I also noticed that the little triangle seems to fit into the big triangle more that twice. I think maybe it would fit in it four times, once on top and three times on the bottom.
- When I *Made a Mathematical Model*, I noticed that the side length of the triangle is twice as big as the side length of the hexagon. I also started thinking about area formulas. There were a lot of quantities in the model.
- When I used *Guess and Check*, it was easy to see what calculations to do to find the area of the triangle, since I had a guess for the perimeter. I wonder how I would find the area of a hexagon if I knew the perimeter?
- I also wonder with *Guess and Check* what I would check. Does the perimeter matter? If I found out what the relationship was for a few different perimeters, would that be enough? Could I use guess and check to find out the relationship in general?

III. Playing with Clues

Clues

The clues we found in the problem were:

- The perimeters are equal.

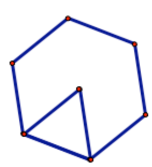
Changing Values

What if the hexagon's perimeter was twice as big (since there are twice as many sides)?

Then the sides would be equal.

What do we know about a triangle and a hexagon with equal sides?

Is the area twice as big because the hexagon's perimeter is twice as big?



Nope, the triangle definitely fits in there 6 times, one for each side. So perimeter ratio is 1:2, the area ratio is 1:6.

Noticing and Wondering: Can we work with the ratios? When the perimeter ratio is 1:1, will the area ratio be 1:3?

Is the little triangle half the area of the equal-perimeter triangle from the original problem? Is it related to that triangle is some way?

We solved this simpler problem with drawing a picture and comparing the triangle with the hexagon, almost like using pattern blocks. Could we solve the original problem with pictures and pattern blocks and tiling?

Ignoring Clues

At first we thought that there was only one clue in the problem: the perimeters are equal. But then we realized there are other facts, which are that the shapes are a hexagon and a triangle, and that the shapes are regular. We decided to think about what would happen if we ignored each type of clue.

○ **Ignore Equal Perimeters**

The shapes could be any size in relationship to each other, so the areas could have any ratio.

○ **Ignore Triangle and Hexagon**

The shapes could be any shape, but they would have the same perimeters. Hmmm…

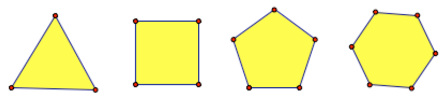
If it was a square and a triangle, they triangle's sides would be 1/3 larger than the triangle's.

The triangle area would be $\frac{base * height}{2}$, and you could find the height with special right triangles.

The square's area would be easy: side * side, which would be 3/4 triangle's side * 3/4 triangle's side. So with a square, it wouldn't be too hard to figure out the ratio.

With a pentagon and a triangle, the pentagon's sides would be 3/5 of the triangle's sides. How would you figure out the area of a pentagon?

Here are some regular shapes with equal perimeters:



○ **Ignore Regular**

If the shapes weren't regular, how could we figure out their areas? Could the areas change a lot if the perimeters were the same? How could we even draw the pictures?

Noticing and Wondering

Without knowing the perimeters are equal and the shapes are regular, it's pretty hard to even start the problem.

We noticed that when we ignored which shapes we were given, we could still figure out the ratios of the side lengths.

We noticed that hexagon and triangle had the only whole number ratio of side lengths of the ones we tried, because the side lengths are multiples of each other.

We noticed that with squares it's easy to use side lengths to find area, but not with hexagons or triangles.

Thinking visually about the problem, squares have a nice relationship with right triangles, but not with equilateral triangles. But hexagons do have some nice relationships with equilateral triangles, like they are made of six equilateral triangles, and many of their angles are multiples of 60 degrees. We wonder if we can use that.

IV. Playing with Pictures and Representations

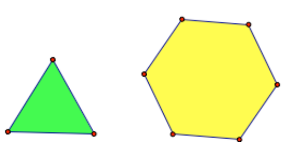
Problem: What is the ratio of the area of the triangle and hexagon

- How many times can the triangle fit inside the hexagon?
- If you broke apart the hexagon and triangle into equal pieces, how many pieces would fit in each one?
- How can you write the hexagon's area in terms of the triangle's area?

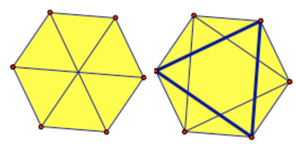
Clue: A triangle and a hexagon.

- A six-sided figure and a three-sided figure.

Clue: Both are regular.

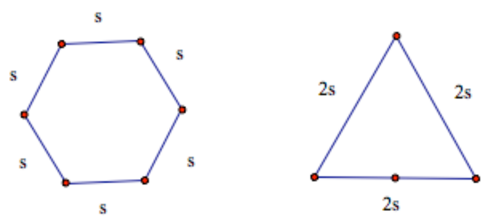


- The angles in the triangle are all 60 degrees, the angles in the hexagon are all 120 degrees, and all of the sides in the triangle are congruent, and all of the sides in the hexagon are congruent.

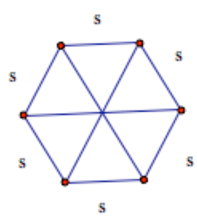


Clue: Both have the same perimeter.

- The hexagon's sides are half the length of the triangle's sides.



- The hexagon is made of 6 equilateral triangles whose sides are half the length of the triangle's sides.



- The triangle is made up of 4 equilateral triangles whose sides are half the length of the triangle's sides.