Geometry PoW Packet

Regional Ratios

April 13, 2009 • http://mathforum.org/geopow/

Welcome!

This packet contains a copy of the problem, the “answer check,” our solutions, teaching suggestions, a problem-specific scoring rubric, and some samples of the student work we received in March 2003, when ***Regional Ratios*** was last used. It is LibraryPoW #2859.

We invite you to visit the PoW discussion groups to explore these topics with colleagues. From the Teacher Office use the link to “PoW Members” or use this URL to go to *geopow-teachers* directly: http://mathforum.org/kb/forum.jspa?forumID=529 [Log in using your PoW username/password.]

The Problem

In ***Regional Ratios***, students find the ratio of the area of a hexagon to that of an equilateral triangle when the perimeters of the two figures are the same. It might be solved by directly calculating the areas using shape-specific area formulas, calculating the areas using the Pythagorean theorem or the properties of 30-60-90 right triangles, or by dissecting the shapes into smaller equilateral triangles.

The text of the problem is included below. A print-friendly version is available from the “Print this Problem” link on the current GeoPoW problem page.

Regional Ratios

A regular hexagon and an equilateral triangle have the same perimeter. What’s the ratio of their areas?

Extra: Find another way to solve it.

Answer Check

The ratio of their areas is 3:2.

If your answer **does not** match our answer,

- is your answer a different form of our answer? (We found the ratio of the area of the hexagon to the area of the triangle. You could do it the other way, since the problem does not specify.)
- did you try starting by assigning a length to the side of the hexagon?
- have you checked any formulas you might have used or arithmetic you might have done?

If any of those ideas help you, you might *revise* your answer, and then leave a *comment* that tells us what you did. If you’re still stuck, leave a *comment* that tells us where you think you need help.

If your answer **does** match ours,

- did you solve it for a specific perimeter? Be sure your answer always works, no matter what the perimeter is.
- do you think there’s an easier way to solve it?
- are there any hints you would give another student?

Revise your work if you have any ideas to add. Otherwise leave us a *comment* that tells us how you think you did—you might answer one or more of the questions above.

Our Solutions

The key math concepts in this problem are perimeter, area, ratios, regular polygons, and “generalization”.

There are three main ways to solve this problem. For each, I'll use 6x as the perimeter of the figures, since that gives "nice" edge lengths of x for the hexagon and 2x for the triangle.

Method 1: Calculate the Areas Using Right Triangles

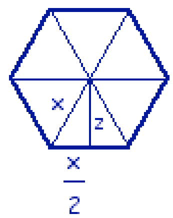
This involves breaking the hexagon into equilateral triangles, finding the area of one of those triangles, and multiplying by 6. For the equilateral triangle, find the height and then find the area. You can find the heights by using the Pythagorean theorem, as I've done below, or by using knowledge of 30-60-90 triangles.

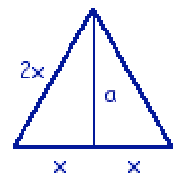
$$z^2 + \left(\frac{x}{2}\right)^2 = x^2$$

$$z^2 + \frac{x^2}{4} = x^2$$

$$z^2 = \frac{3x^2}{4}$$

$$z = \frac{x\sqrt{3}}{2}$$





$$x^2 + a^2 = (2x)^2$$

$$x^2 + a^2 = 4x^2$$

$$a^2 = 3x^2$$

$$a = x\sqrt{3}$$

$$\text{area} = 6\left[\frac{1}{2} * x * \frac{x\sqrt{3}}{2}\right]$$

$$\text{area} = 6\left[\frac{x^2\sqrt{3}}{4}\right]$$

$$\text{area} = \frac{3x^2\sqrt{3}}{2}$$

$$\text{area} = \frac{1}{2} * 2x * x\sqrt{3}$$

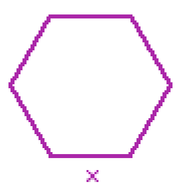
$$\text{area} = x^2\sqrt{3}$$

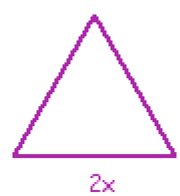
$$\text{ratio} = \frac{\frac{3x^2\sqrt{3}}{2}}{x^2\sqrt{3}} = \frac{3}{2}$$

Note: A lot of students will this method, but will not find the height of the triangle. They just use one side as the base and one as the height. They do end up with the right answer, but as this shows a vast lack of understanding of the ideas of base and height, these students should be scored as an apprentice in Interpretation, since they do not understand one of the key concepts in the problem.

Method 2: Direct Area Formulas

Another way to solve this is to look up the formulas for the areas of a regular hexagon and an equilateral triangle.





$$\text{area} = \frac{3a^2\sqrt{3}}{2}, \text{ where } a = x$$

$$\text{area} = \frac{3x^2\sqrt{3}}{2}$$

$$\text{area} = \frac{a^2\sqrt{3}}{4}, \text{ where } a = 2x$$

$$\text{area} = \frac{(2x)^2\sqrt{3}}{4}$$

$$\text{area} = \frac{4x^2\sqrt{3}}{4}$$

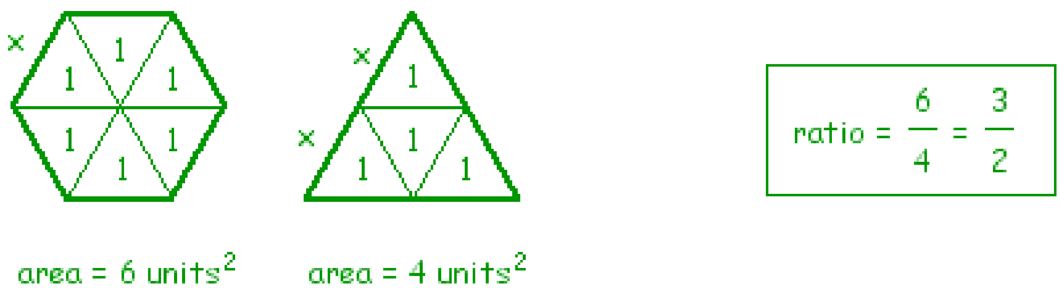
$$\text{area} = x^2\sqrt{3}$$

$$\text{ratio} = \frac{\frac{3x^2\sqrt{3}}{2}}{x^2\sqrt{3}} = \frac{3}{2}$$

Method 3: Dissection

A third way to solve this is to break both the hexagon and the equilateral triangle into their component parts, or smaller equilateral triangles. Since all of these equilateral triangles are congruent, you can get the ratio of the areas right from there.

The small rectangles will be congruent because in each case we start with a regular figure that can be composed of equilateral triangles.



Method 4: Trigonometry

You could also use trigonometry, though that is likely to lead to approximations and inaccurate answers. Too much rounding can lead to an inaccurate answer (though the strategy may well be sound). It could work out if exact numbers, such as √3/2 for sin60, are used.

Teaching Suggestions

If students need help getting started, you might suggest that they pick a perimeter, any perimeter. It will be the same for both shapes, so now they can figure out the side length of each shape. (They may then revise their initial choice so that it's divisible by 6.) While the problem should ultimately be solved using a general perimeter (such as 6p), it doesn't hurt to start by using a specific perimeter.

One thing to watch for is not in any way unique to this problem – watch for kids who aren't using the height in area formulas that call for the height. The first time we used this problem, many kids used the side length as both the base and height when finding the area of an equilateral triangle. The reason you need to watch carefully in this situation, however, is that students can get the right answer anyway. Since we're finding a ratio of areas, and not actual areas, if you use the same wrong formula or wrong height, you can still get the correct relative size of the areas.

Ratios may also trip some students up. For starters, you might just remind them that a ratio is a comparison, and see if that helps them get going with way to find the areas, or at least compare the areas.

You can find a few online resources to go along with this problem at

http://mathforum.org/geopow/puzzles/supportpage.ehtml?puzzle=416

All the resources to go with this and all other Current Geometry PoWs from this year are linked to from

http://mathforum.org/pow/support/

Sample Student Solutions

Focus on Strategy

In the solutions below, we've focused on students' strategies. Generally speaking, this reflects whether the student has picked a sound method with which to solve the problem. In this case, it might include calculating the areas of the figures using the area of a triangle, calculating the area of each shape using the formula for the area of that shape, or by dissecting the shapes into smaller congruent triangles from which the ratio can be calculated.

Vanessa
age 15

Strategy
Novice

ap(hexagon):bh (triangle)

the area of a equilateral triangle is 1/2bh (b is base and h is height) and the area of a regular hexagon is 1/2ap (a is apathum and p is perimeter)

Vanessa knows she needs to find areas, and she's got some formulas that will work. But it seems like she doesn't know what to do without any numbers to work. I might suggest that she just pick a number for the side of the hexagon and see if that helps her get started.

Jeff
age 15

Interpretation
Novice

the ratio is 3:1

I could not find in the book how to find the ratio. I am taking a guess

It is not so uncommon for a student to assume that there is "a way" to do everything. I would start out by suggesting a side length of 1 for the hexagon, and asking him if he can find the area (he will likely be able to find a formula for that in the book).

Jen
age 15

Strategy
Apprentice

The ratio of the areas of the hexagon to the triangle is 1:1.483.

From previous experience, I knew that all hexagons are broken up into six equilateral triangles. This knowledge would later help me find the area of a few "practice" hexagons that would allow me to compare the area of a triangle to the area of a hexagon because I didn't know the real formula for the area of a hexagon.

So, the first thing I did was to draw a few hexagons and triangles and compare the areas. I couldn't use just any sizes for the hexagon and the triangle; they had to be proportionate to the ones in the problem. I found the ratio of the side lengths in order to make the sides proportionate. The ratio of the sides from the hexagon to the triangle is 2:1. I found this out because the triangle and the hexagon have the same perimeter, but a different number of sides. A hexagon has 6 and a triangle has 3. When reduced this fraction is 2:1, meaning two of the hexagon's sides must add up to 1 of the triangles sides in order to have the same perimeter.

Jen clearly understands how the triangle and hexagon are related, and has figured out appropriate side lengths that make the perimeters equal. She's also done a good job finding the ratio. But then it sounds like she has drawn some pictures and measured them. I would ask Jen if she knows a way to calculate the height of an equilateral triangle when you know the length of the base.

I found the area of all the “practice” triangles by multiplying 1/2base X height. As I said before, a hexagon is broken up into six triangles. To find the area of the “practice” hexagons I found the area of one of the triangles within the hexagon and then multiplied it by six.

All of these figures helped me out because they allowed me to see what they all had in common. I found that when I divided the area of the hexagon by the area of the triangle, they all equaled 1.483. I formed my ratio around this number. I found that 1 hexagon is equal to 1.483 triangles.

Katie
age 16

Strategy
Apprentice

The ratio of the area of a regular hexagon to an equilateral triangle with the same perimeters is 1:4.

In order to solve this I found first the area of the triangle using a concrete example of 6 for the sides of the triangle. The equation that i used was (s^2/4)*root 3. This answer was 9 root 3. The I found the area of the hexigon using a concrete example of 3 for each side. The formula that i used was 6(s^2/4)* root 3. My answer for this was (9/4) root 3. Then I divided the area of the hexigon into the area of the triangle. My answer was 1/4. Therefore, the ratio of the area of a regular hexigon with the area of an equilateral triangle whose perimeters are the same is 1:4.

Katie knows the formulas for the area of each of the figures, and she’s picked appropriate side lengths (they’ll result in equal perimeters). But she has made an arithmetic mistake in finding the area of the hexagon. I would ask her about that, but then also ask her if she can show that it will always work this way, or if it is specific to the side lengths she chose, trying to move her in the direction of a general solution.

Rebecca
age 14

Strategy
Apprentice

The ratio of the area of the hexagon to the area of the triangle is 3:2.

1) First, let us plug numbers into the problem. Allow each of the hexagon's sides to have a length of 4, and each of the triangle's to have a length of 8.

2) In order to find the area of the hexagon, we can draw its three diagonals to divide it into 6 equilateral triangles.

3) Use the Pythagorean theorem to find the height of each triangle:

2^2 + 4^2 = c^2

4 + 16 = c^2

c = 2 root 5

4) Area of a triangle = (b x h)/2, so area = (4 x 2 root 5)/2 = 4 root 5

5) Since there are six little triangles: 4 root 5 x 6 = 24 root 5

6) Now, to find the height of the big triangle:

4^2 + 8^2 = c^2

16 + 64 = c^2

c = 4 root 5

7) Area = (8 x 4 root 5)/2 = (32 root 5)/2 = 16 root 5

8) Ratio = 24 root 5 : 16 root 5

Rebecca has done a very nice job using the Pythagorean theorem to find the height of an equilateral triangle and the resulting area of the equilateral triangles. She hasn’t simplified her final answer, but that’s easily encouraged (and likely easily done). However, she’s used a specific edge length. So I would ask her, as with Katie, if she can find a way to show that this will always work, regardless of the actual lengths.

Kyle and Lee
age 16

Strategy
Apprentice

The ratio of the areas hexagon:triangle would be 3:2.

In an example problem we set the perimeter to 30 and then found the area of each shape and then reduced them down.

Triangle	Hexagon
Area=1/2base*height	Area=1/2perimeter*altitude
A=1/2(10*10)	A=1/2(30*5)
A=1/2(100)	A=1/2(150)
A= 50	A=75

So the ratio of hexagon:triangle would be 75:50 then you can reduce that to 3:2 by dividing both sides by 25.

Kyle and Lee have done a nice clear job of calculating the area of each figure, and they've arrived at the correct ratio. However, they've used the same number for the base and the height of each triangle. I would point that out to them and ask them if they know a way to calculate the actual height before continuing with their area calculations.

Mason
age 14

Strategy
Practitioner

The ratio of the areas of a hexagon and a triangle that have the same perimeter is 2/3.

If the perimeter of the triangle is P, each side of the triangle would be P/3. The height of the triangle would be sqrt[(P/3)^2-(P/6)^2], which eventually comes out to (P sqrt 3)/6. When the height is multiplied by the base and divided by 2, (P sqrt3)/6 * P/3 * 1/2, the answer is the area of (P^2 sqrt3)/36.

The formula to find the area of a hexagon is A=aP/2. One side of the hexagon is P/6. The triangle formed here is a 30, 60, 90, triangle, so if the base is P/12, the apothem (a) is (P sqrt3)/12 (12 sqrt3)/3= Pa simplified into (4 sqrt3)/P=a. Plugged into the formula A=aP/2, the area is A= (P^2 sqrt3)/24.

The area of the triangle is (P^2 sqrt3)/36 and the area of the hexagon is (P^2 sqrt3)/24. Therefore, the ratio of their areas, Area of HEX. / Area of TRI. is [(P^2 sqrt3)/36]/[(P^2 sqrt3)/12], simplified to 24/36 which equals 2/3.

Mason has a solid solution, using side length and the apothem. I would probably make some suggestions that would help him make his work, and his algebra especially, a little easier to read. I might also challenge him to find another way to solve the problem.

Patrick
age 13

Strategy
Practitioner

The ratio of the area of the hexagon to the area of the triangle is 3/2.

First we know that the perimeter of both figures is the same. Label this value 3x. That means that, since all sides of the equalateral triangle (figure A) are congruent, each side is (3x/3) which is x. For the same reason, the regular hexagon (figure B) has a side of (3x/6)=(x/2). Since the angles of any regular n-gon is (180(n-2))/n, the measure of each angle of the hexagon is (180(6-2))/6 = (180*4)/2 = 720/6 = 120.

Now, draw all three diagonals of the hexagon. Each diagonal must bisect the angles, so the measures of the outside angles of the small triangles formed are all 60. since 180-60-60=60, the angles of the small triangles that are closest to the center are 60. Therefore, the small triangles are equalangular and therefore equalateral. Since every side of the hexagon is (x/2), every side of every equalateral triangle inside the hexagon must be (x/2). By ASA, every small triangle inside the hexagon is congruent, so we only have to look at one to find the area (Figure C).

Now we have two equalateral triangles: one with side x (Fig A) and one with side x/2 (Fig C). The area of an equalateral triangle is ((s)^2*(sqrt3))/4, where s is the side of the triangle. This is because the altitude of an equalateral triangle forms a 30-60-90 triangle. Therefore the altitude is s*(sqrt3)/2. The area is then s*s*(sqrt3)/2/2 or (s^2*(sqrt3))/4. The area of the equalateral triangle with side x (Fig A) is (x^2*(sqrt3))/4 which does not simplify. The area of the small triangle with side x/2 (Fig C)is (((x/2)^2*(sqrt3))/4) = ((x^2*(1/4)*(sqrt3))/4) = ((x^2*(sqrt3))/16).

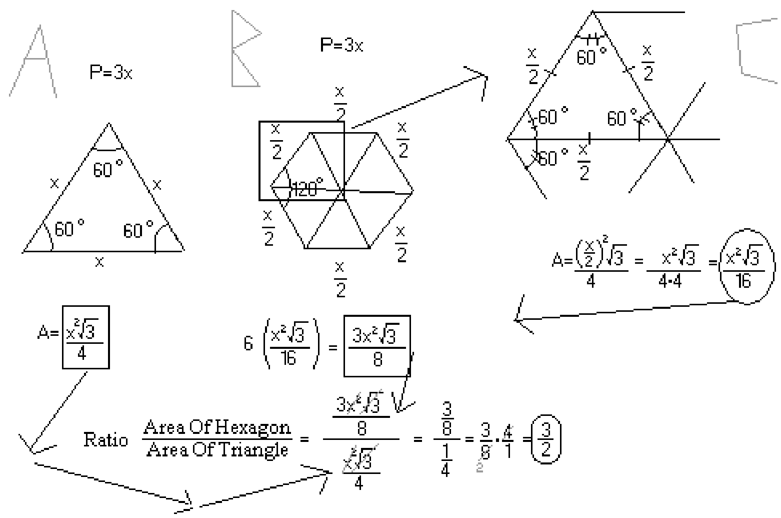
Patrick has left pretty much no stone unturned here! He has even included a picture. Even more so than Mason, above, Patrick would most benefit from some tips about how to more clearly present math in plain text, since this is pretty dense. I would also challenge him to find another way to solve it – I wonder what he would think of the dissection solution, after doing all this work?

Finally we get to the hexagon (Figure B). Since the hexagon is the union of the six congruent equalateral triangles, the area of the hexagon equals 6 times the area of one small triangle. That means that the area of the hexagon equals 6((x^2*(sqrt3))/16)=((3*x^2*(sqrt3))/8).

Now we get to the ratio part:

The ratio of Area of the hexagon to the Area of the triangle is ((3*x^2*(sqrt3))/8)/((x^2*(sqrt3))/4), which equals (3/8)/(1/4) because the sqrt3 cancels out and the x^2 cancels out. (3/8)/(1/4)=(3/8)(4/1) because of the way that one divides fractions. That value is (3*4)/(8*1) which equals 12/8 and simplifies to 3/2.

Therefore, the final answer is 3/2!



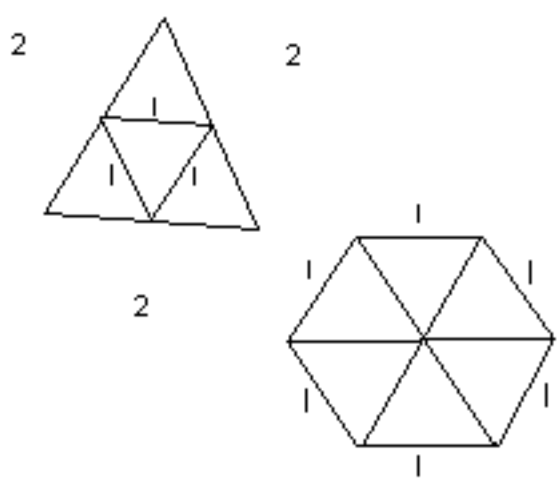
Maria
age 15

Strategy
Practitioner

The ratio of the area of the triangle to the hexagon is 1:1.5

We have a triangle and a hexagon that have a perimeter of 6 units. The equilateral triangle has sides of 2 and the regular hexagon has sides of 1. Considering that in an equilateral triangle with a sidelenghths of two there fits exactly 4 triangles with side lenghts of 1, and that in a regular hexagon with a perimeter of 6 there fits six equilateral triangles with side lenghths of 1, then you can conclude that six equilateral triangles with side lenghths of 1 will fit in a regular hexagon. Therefore one and a half equilateral triangles with side lenghts of 2 will fit into a reguar hexagon with side lenghts of 1.

Maria has provided a very helpful diagram along with her solution. While she uses a specific length, she doesn't really use it in her solution, other than as a comparison. I would probably ask her if it matters, and whether she could change her solution to be more general in that way.



Genia
age 11

Strategy
Practitioner

The ratio of their areas 3:2 (hexagon:triangle).

Consider that one of the sides of the hexagon is x. (it's regular, so it doesn't matter which one) Then, the sides of the triangle are 2x, because you have the same perimeter with half the sides to divide it. It the perimeter is thought of in terms of x: 6x , for the hexagon it would be divided by 6, and you would have x. But for the triangle, you would divide by 3, and have 2x.

Now think of the two shapes as consisting of equilateral triangles. All of the triangles are congruent. Their sides are all x (think of the way to find the area of a hexagon: you divide it into 6 triangles).

On the hexagon, each of the triangles has one side that is one of the six sides of the hexagon.

```
     x
  *------*          **
x* *   * *x       x*  *     Figure not drawn to scale
*---* *---*       *----*2x  See how the two x's in the triangle's side
x*  * *  *x     x* *  * *   are two sides of two congruent triangles?
  *------*      *---**---*
     x            x    x
```

You see that one can be filled with exactly 6 triangles, the other -

with only 4 of the same exact triangles. So, their areas are 6:4. In

lowest terms, that's 3:2.

Genia put some serious work into that picture! She has also solved it for any side length. I would probably ask her to say a little bit about how she knows all the triangles are congruent, as she claims, and ask her if she thought of any other ways to solve it.

Scoring Rubric

On the last page is the **problem-specific rubric,** to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work. A **generic student-friendly rubric** can be downloaded from the *Scoring Guide* link on any problem page. We encourage you to share it with your students to help them understand our criteria for good problem solving and communication.

We hope these packets are useful in helping you make the most of Geometry PoWs. Please let me know if you have ideas for making them more useful.

~ Annie

Geometry PoW Scoring Rubric for *Regional Ratios*

For each category, choose the level that *best describes* the student's work

	Novice	Apprentice	Practitioner	Expert
Problem Solving				
Interpretation	does one or none of the things listed under Practitioner	does only two of the things listed under Practitioner	understands that the two figures have the same perimeter and that they're regular seems to understand how to find a ratio attempts to find the ratio of the areas (can be hexagon:triangle or triangle:hexagon)	is at least a Practitioner in Strategy and has solved the Extra question
Strategy	has no ideas that will lead them toward a successful solution	has a strategy that somehow relies on luck finds a ratio of the areas using a specific edgelength without explaining why that doesn't matter	has a strategy that relies on sound reasoning, not luck solves the problem for any perimeter, not just a specific one (unless they talk explicitly about why it doesn't matter and will result in the same solution regardless) might split both figures into smaller equilateral triangles (4 for the triangle, 6 for the hexagon) might calculate the areas after assigning lengths to the edges	Expert is often achieved by using two different strategies, but that's covered by the Extra question (and hence reflected in Interpretation), so maybe three different methods?
Accuracy	has made many errors	has made several mistakes, or has used vocabulary incorrectly	makes no mistakes of consequence and uses largely correct vocabulary and notation	[generally not possible]
Communication				
Completeness	has written almost nothing that tells you how they found their answer	shows work without explanation, or gives an explanation without showing any work	shows and explains the steps taken and why they are reasonable steps, which might include: • any area formulas used • what it means to find a ratio • any variables assigned as edgelengths and why the edgelength of the triangle is twice that of the hexagon	includes additional helpful information, doesn't just add more for the sake of adding more
Clarity	explanation lacks clarity and organization	explanation is difficult to follow length warrants separation into more paragraphs lots of spelling errors/typos	explains the steps that they *do* explain so that another student would understand (needn't be complete to be clear) makes an effort to check their formatting, spelling, and typing (a few errors are okay)	answer is clearly written and well-organized formats things very clearly
Reflection	*The items to the right are considered reflective, and could be in the solution or their comment:*	checks their answer (not the same as viewing our "answer check") reflects on the reasonableness of their answer	connects the problem to prior knowledge or experience explains where they're stuck summarizes the process they used	comments on and explains the ease or difficulty of the problem **revising their answer and improving anything**
	does nothing reflective	does one reflective thing	does two reflective things	does three or more reflective things or great job with two

