The Math Forum: Problems of the Week

Problem Solving and Communication Activity Series

Round 17: Playing

When students do the *Noticing/Wondering* activity, we often have them try to group their noticings into "quantities" and "relationships." With a little practice, students get adept at finding the quantities and the relationships that are explicitly stated in the problem. However, interesting math problems usually have deeper layers of relationships that only emerge as problem solvers "play" with the relationships and quantities.

In the recent activities focusing on *Planning* and *Getting Unstuck*, we began to highlight some of the phases of problem solving, and to show how many of the activities in this series can be used to explore relationships as you begin problem solving or if you get stuck along the way.

Continuing in this vein, *Round 17* focuses on some of the ways problem solvers play with relationships and explore patterns before they delve deeply into a single problem-solving strategy. In order to make clear different aspects of problem solving, we've broken the "play" process out somewhat artificially – expert problem solvers move back and forth fluidly between understanding the problem, playing with relationships, and carrying out strategies. However, for purposes of illustration, we think it will be useful to focus on those phases separately.

The activities are written so that you can use them with problems of your choosing. There is no sequence to the activties. Select one or more that seem appropriate or adaptable to your classroom. We include a separate section after the activity descriptions to provide examples of what it might look like when students apply these activities to the current Math Fundamentals Problem of the Week.

Problem-Solving Goals

Good problem solvers:

- Play and explore as they solve problems.
- Look for deeper and hidden relationships.
- Try to uncover more and more interesting math.
- Try multiple approaches or ways of looking at a single problem.

Communication Goals

Problem solvers use communication as they play to help them:

- Keep track of interesting things they noticed and wondered.
- Represent the problem in new ways.
- Paraphrase the problem.
- Share their own perspectives and ideas and learn from others.

Activities

I. Calculating (and Noticing) as you Go

Sample Activity: Calculate as you Go

Format: Students working in pairs.

When you see quantities in the problem, you may not see how to solve the problem, but you might think of some calculations you could do. Try doing some of these calculations that come into your head, even if you don't know that they will help you solve the problem.

Be sure to tell your partner:

What you did (what quantities and operations you used).

What the units of the results are (what you are counting or measuring).

As you calculate, notice if you get any interesting results, or if any of the calculations seem particularly helpful.

II. Playing with Strategies

Sample Activity: Speed Dating

Format: students working in groups of three to five.

Materials: strips of paper, pens or pencils, loose-leaf paper.

In order to get the juices flowing and begin to investigate and unearth more relationships, it can be helpful to try a lot of different ideas quickly. See what you notice, but don't get too bogged down in one idea.

1) Each person writes a strategy or short description of something to try on a strip of paper. The activity will be more fun if each person chooses a different strategy. Some good examples: *Guess and Check, Change the Representation, Tables and Patterns,* and *Solve a Simpler Problem*.
2) When everyone is ready, each member of your group should pass his/her strip of paper to the left. You have three minutes to do what you can with the strategy or idea that you received. Write your work and what you notice and wonder on your own sheet of loose-leaf paper (this way, at the end, you will have ideas from a few different strategies that you can look back at as you work on the problem).
3) After three minutes, stop wherever you are and draw a line or a box around your work, and write the name of the strategy used.
4) Pass the strategy strip you were working on to your left, and receive a new one on your right. You have three minutes to work on the new strategy. Keep working on your own paper with a new section for each strategy that you do.
5) Pass papers every three minutes until you receive the strategy strip that you originally wrote. Finish by working on that strategy for three minutes.
6) As a group, add any new relationships, patterns, quantities, interesting ideas, or things you are wondering about to your list of noticings and wonderings.

Key Outcomes

- Get a better understanding of the problem by playing with a variety of ideas before solving the problem.
- Identify and pull together the most promising solution path from multiple representations or multiple strategies.
- Discover and note deeper, hidden relationships that emerge as you play with various possibilities.

III. Playing with Clues

Sample Activity: What If…

Format: students working in groups of three to five.

"Clues" is a useful shorthand for the longer phrase "quantities and relationships you noticed or wondered about."

One way to understand how a particular clue is a useful part of the problem is by changing it and noticing how the problem changes. I might change the value of a clue, or even pretend I don't know it. My goal is to focus on what changes in the problem, and how I can use that information to understand the clue better.

1) With your group, go through each clue or set of clues. Write down ways you could change the value of quantities in that clue or the constraint that it imposes.
2) For each clue, play out the problem a little bit with the new value(s). How would the problem change? Would it be easier or harder? What would be easier or harder about it? Would the results be different?
3) Now go through the clues again, this time ignoring one clue or set of clues at a time, pretending that information was never given.
4) How does the problem change when those clues are ignored? Does it make a simpler version you can use to learn more? Does it change the number of possible answers? What else changes?
5) After you've gone through all the clues, look back at the original problem. What new understanding have you gained? Do you see more uses for any of the clues? Do any of the clues seem more necessary (or unnecessary)?

Key Outcomes

- Explore how the problem was constructed.
- Generate additional information and perspective by changing or ignoring clues.
- Gain better understanding of the problem by thinking about simpler (and harder) versions.

IV. Playing with Pictures (and Representations)

Sample Activity: How Else Can We Say It?

Format: students working in groups of three to five (depending on the number of clues in the problem).

Sometimes the clues are said one way, but if you said them (or wrote them or drew them) just a little differently, you would see different relationships. In this activity, try to express the clues as many different ways as you can.

1) Each person picks one clue from the problem and rewrites it or draws a picture of it or somehow changes *how* it's said without changing *what* is said.
2) After a few minutes, each person passes the clue to their left. Try to add another way to say (or draw or represent) the clue, adding to those already written down.
3) Keep passing clues to the left as long as you can come up with new ways to express them.
4) Once you've run out of ideas working individually, hold a group discussion:
    a. Check if any of the different expressions changed *what* the clues really meant. In this activity you don't want to change what the clues mean. You just want to get a new perspective on what they mean.
    b. Did any new information or perspective emerge that helps you see an approach to solving the problem?

Key Outcomes

- Understand the problem better using multiple representations of key information.
- See the problem from a fresh perspective.

Examples: She Counts Seashells (FunPoW)

The goal of these lessons is for the students to reflect on their own process in exploring the information given in a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and instead focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "funpow-teachers" discussion group (http://mathforum.org/kb/forum.jspa?forumID=526) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

I. Calculating (and Noticing) as you Go

Partnership 1

Student 1: Here are some calculations I can think of:

Counting by twos: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24…

Student 2: I get the point! Are you counting seashells?

Student 1: No. The number of seashells has to have one left over. So there could be 25, but not 24.

Student 2: So the number of seashells could be 1 or 3 or 5 or 7... all of the odd numbers.

Student 2: What calculations can we do with the counting by threes?

Student 1: Well, counting by threes means 0, 3, 6, 9, 12, 15, 18, 21, 24

Student 2: For the number of seashells you have to have two left over this time.

Student 1: So if you start at two and count by threes, do you get the possible numbers of seashells? 2, 5, 8, 11, 14, 17, 23, 26…

Student 2: Yeah, those sound right. How did you calculate them?

Student 1: Well, first I started by going by threes each time (2, 2+3, 5+3). Then I noticed each number is one less than the next multiple of three. So I did 12-1 = 11, 15-1 = 14, 18-1 = 17.

Student 2: Let's see if that works when we count by fours. We can see if making a list of all the numbers one less than a multiple of four gives us the "three left over" numbers.

Student 1: 4-1 = 3, 8-1 = 7, 12-1 = 11, 16-1 = 15, 20-1 = 19, 24-1 = 23, 28-1 = 27.

Student 2: So the list is: 3, 7, 11, 15, 19, 23, 27.

Student 1: That's the same as starting at 3 and going up by 4s, so it's the "three left over" numbers.

Partnership 2

Student 3: I don't know what calculations to do, since I don't know how many shells there are. If I knew how many shells there were, I would count them by twos, threes, fours, etc. and see what was left.

Student 4: Maybe we could guess how many shells there are. What I wonder is what would make a good guess? What numbers would have the right amount left over when you divided by 2 and 3 and 4 and 5?

Student 3: I know all the odd numbers would have 1 left over if you divide them by 2.

Student 4: I know how to tell if a number is divisible by 3. You see if the digits add up to a number that three goes into. So to check 99 you would do 9+9 = 18. Does 3 go into 18? Yes, but even if I didn't know, I can do the same test with 1+8 = 9, and 3 definitely goes into 9.

Student 3: Slick. So 3 goes into 99, which means there will be none left over and that number wouldn't work.

Student 4: I wonder if we could use some of the other divisibility tricks to tell what other numbers are good guesses. I don't know any tricks for four, but there are a lot for five.

II. Playing with Strategies

Possible Strategies

- Guess and Check
- Act it Out with counters
- Work Backwards

Results

- **Guess and Check (3 minutes):**

  Guess the number of seashells she has.

  Guess 23, since I know 2, 3, 4, and 5 will have some left over.

  Count by 2's: 2,4,6,8,10,12,14,16,18,20,22 – 1 left over (good!)

  Count by 3's: 3,6,9,12,15,18,21 – 2 left over (good!)

  Count by 4's: 4,8,12,16,20 – 3 left over (good!)

  Count by 5's: 5,10,15,20 – 3 left over (dang!)

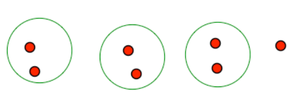
  Next guess: 24 (I know it will work for 5's)

  Count by 2's: 2,4,6,8,10,12,14,16,18,20,22,24 – 0 left over (dang!)

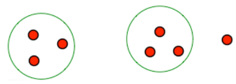
  Next guess: TIME'S UP

- **Act it Out with counters (3 minutes):**

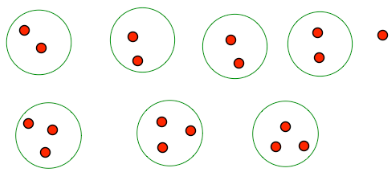
  I counted out some counters, and circled groups of 2 and made sure I had one extra. I was going to keep adding groups of 2, but I didn't know when to stop, so I went on to the next clue.



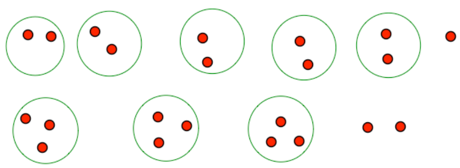
  I put 7 counters in groups of 3:



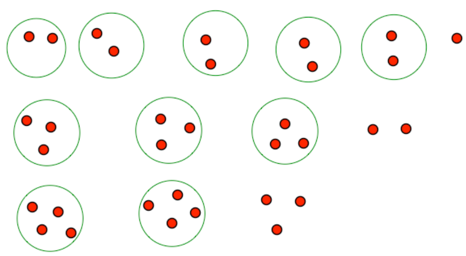
But then there wasn't 2 left over, so I added two more to both rows:



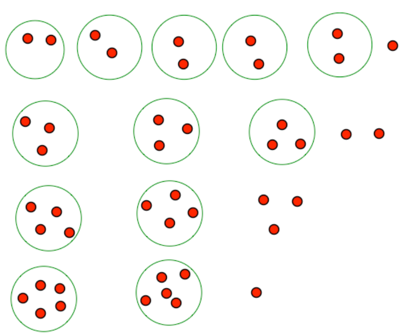
But then there weren't any left over for counting by 3's, so I added 2 more again:



Having 11 counters worked for 2's and 3's. I decided to add in the 4's.



Having 11 counters worked for 2's, 3's, and 4's. I decided to add in the 5's.



There was only 1, not 4, left over for the 5's, so I will add more counters to each row. I added by 2's, because I know the top row will work if I keep adding 2's. I have to add enough 2's so that I get 3 more left over when I count by 5's.

TIME'S UP – I didn't get to add the next set of 2's.

- **Work Backwards**

  This is a hard one – let's see what I can do… I don't know how many shells she ended up with… What are the steps in the problem?

  First, she counts by twos, but then has one left over. Maybe "backwards" would be starting with the one left over and adding twos? Or maybe starting at the end and taking away twos until you have one left over?

  Then she counts by threes, and has two left over.

  Then she counts by fours, and has three left over.

  Last, she counts by fives and has four left over.

  One way to work backwards would be to start with the clue about counting by fives. I could think about counting backwards by fives and ending up at four. TIME'S UP!

New Noticings and Wonderings

- From *Working Backwards* we noticed that we could work with the clues in different orders. Maybe some clues are more helpful than others.
- We also noticed that we could think about counting forwards or backwards by twos, threes, fours, and fives, and seeing where we start or where we end up.
- From *Acting it Out* we noticed that to make the counting by twos work we always had to add two more counters, and that sometimes that result gave us good results for the counting by threes, and sometimes not.
- We wonder if there's a rule for how many times to add two to make the twos and threes work at the same time.
- From *Guess and Check* we noticed that some numbers work for almost all of the clues.
- We also noticed that even numbers don't work, because of the twos.
- We also noticed that 23 didn't work for the fives because 23 was only 3 away from 20, not 4 away. 24 would have worked, but it's even and we need an odd number.
- We wonder if the next number that would have worked for counting by fives and by twos would be 29. The next number counting by fives after 20 is 25, and four more than that is 29. And then we wonder if 39 is the next number that works for counting by fives and by twos again.

III. Playing with Clues

Clues

The clues we found in the problem were:

- When she counted them by 2s, there was 1 left.
- When she counted them by 3s, there were 2 left.
- When she counted them by 4s, there were 3 left.
- When she counted them by 5s, there were 4 left.

Changing Values

We decided to first change all of the clues so that the number left over was the same. We thought that would make the problem easier.

If you count by twos, threes, fours, and fives and always have one left over, then you always have one more than the twos, threes, fours, or fives times tables. Watch:

Twos	Seashells	Threes	Seashells	Fours	Seashells	Fives	Seashells
2	3	3	4	4	5	5	6
4	5	6	7	8	9	10	11
6	7	9	10	12	13	15	16
8	9	12	13	16	17	20	21
10	11	15	16	20	21	25	26
12	13	18	19	24	25	30	31
14	15	21	22	28	29	35	36

Then you have to look for a number that's in all of seashells columns, to find one that works for each clue. Or you can look for a number that is in each of the multiples columns. 13 is in all of the seashell columns except for the last one. This goes with 12, which is in each of the multiples columns except for the fives. Once you find a number that is a multiple of all, then you can just add one more to find the seashell number.

Noticing and Wondering: When there was an easy pattern to the clues (such as one shell left over every time), we could easily use multiples to find the answer.

In our problem, the number of seashells left over is different each time, but there is a system to it. Each time, the number of seashells left over is one less than the number you're counting by.

We think that since there is a pattern in the clues, even though it's not a simple pattern, we might be able to use multiples to solve the problem. We could make a table and look for patterns.

Ignoring Clues

We decided to focus on just the clue “When she counted them by 2s, there was 1 left,” and see what the answers could be.

There are many, many possible answers! Any odd number would have one left over when you count by twos. All alone, that clue isn’t very helpful…

We thought about going through each clue that way, but then we wondered about what happened when you looked at just a few clues in combination. So we thought about what would happen if you just had the counting by twos clue and the counting by threes clue.

We know that any odd number works for counting by twos. Which possibilities that work for counting by threes are odd?

The numbers that work for counting by threes are: 2, 5, 8, 11, 14, 17, etc.

Every other one is odd. So the next odd ones would by 23, 29, 35, etc. We added 6 each time instead of 3.

Noticing and Wondering

When we ignored more clues there were more possible answers. As we added clues back in, we started skipping possible answers.

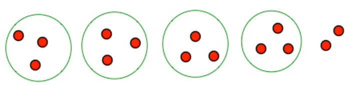
We noticed that when we thought about clues for counting by twos and threes, we had to count by sixes to find possible answers. We wonder if when we look at counting by twos, threes, and fours, we will count by 24s (2*3*4 = 24).

IV. Playing with Pictures

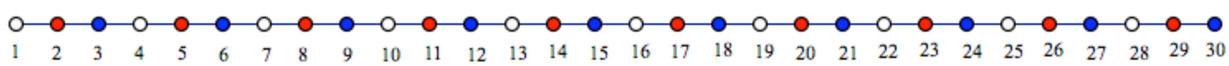
Note: only one clue is illustrated here, for purposes of brevity.

Clue: “When she counted them by threes, there were two left.”

- The number of seashells is not divisible by three: there is a remainder of two.
- You could draw the seashells counted by threes like this:



- The number of seashells could be numbers like 2, 5, 8, 11, 14, 17, 20, etc.
- The number of seashells could be any of the red-dotted numbers below. The multiples of three are blue-dotted.



- There are two ways we could say each clue: For example:

    The number of seashells is two more than a multiple of three.

    The number of seashells is one less than a multiple of three.