



The Math Forum: Problems of the Week

Problem Solving and Communication

Activity Series

Round 17: Playing

When students do the *Noticing/Wondering* activity, we often have them try to group their noticings into “quantities” and “relationships.” With a little practice, students get adept at finding the quantities and the relationships that are explicitly stated in the problem. However, interesting math problems usually have deeper layers of relationships that only emerge as problem solvers “play” with the relationships and quantities.

In the recent activities focusing on *Planning* and *Getting Unstuck*, we began to highlight some of the phases of problem solving, and to show how many of the activities in this series can be used to explore relationships as you begin problem solving or if you get stuck along the way.

Continuing in this vein, *Round 17* focuses on some of the ways problem solvers play with relationships and explore patterns before they delve deeply into a single problem-solving strategy. In order to make clear different aspects of problem solving, we’ve broken the “play” process out somewhat artificially – expert problem solvers move back and forth fluidly between understanding the problem, playing with relationships, and carrying out strategies. However, for purposes of illustration, we think it will be useful to focus on those phases separately.

The activities are written so that you can use them with problems of your choosing. There is no sequence to the activities. Select one or more that seem appropriate or adaptable to your classroom. We include a separate section after the activity descriptions to provide examples of what it might look like when students apply these activities to the current Algebra Problem of the Week.

Problem-Solving Goals

Good problem solvers:

- Play and explore as they solve problems.
- Look for deeper and hidden relationships.
- Try to uncover more and more interesting math.
- Try multiple approaches or ways of looking at a single problem.

Communication Goals

Problem solvers use communication as they play to help them:

- Keep track of interesting things they noticed and wondered.
- Represent the problem in new ways.
- Paraphrase the problem.
- Share their own perspectives and ideas and learn from others.

Activities

I. Calculating (and Noticing) as you Go

Sample Activity: Calculate as you Go

Format: Students working in pairs.

When you see quantities in the problem, you may not see how to solve the problem, but you might think of some calculations you could do. Try doing some of these calculations that come into your head, even if you don’t know that they will help you solve the problem.

Be sure to tell your partner:

What you did (what quantities and operations you used).

What the units of the results are (what you are counting or measuring).

As you calculate, notice if you get any interesting results, or if any of the calculations seem particularly helpful.

II. *Playing with Strategies*

Sample Activity: Speed Dating

Format: students working in groups of three to five.

Materials: strips of paper, pens or pencils, loose leaf paper.

In order to get the juices flowing and begin to investigate and unearth more relationships, it can be helpful to try a lot of different ideas quickly. See what you notice, but don't get too bogged down in one idea.

- 1) Each person writes a strategy or short description of something to try on a strip of paper. The activity will be more fun if each person chooses a different strategy. Some good examples: *Guess and Check*, *Change the Representation*, *Tables and Patterns*, and *Solve a Simpler Problem*.
- 2) When everyone is ready, each member of your group should pass his/her strip of paper to the left. You have three minutes to do what you can with the strategy or idea that you received. Write your work and what you notice and wonder on your own sheet of loose-leaf paper (this way, at the end, you will have ideas from a few different strategies that you can look back at as you work on the problem).
- 3) After three minutes, stop wherever you are and draw a line or a box around your work, and write the name of the strategy used.
- 4) Pass the strategy strip you were working on to your left, and receive a new one on your right. You have three minutes to work on the new strategy. Keep working on your own paper with a new section for each strategy that you do.
- 5) Pass papers every three minutes until you receive the strategy strip that you originally wrote. Finish by working on that strategy for three minutes.
- 6) As a group, add any new relationships, patterns, quantities, interesting ideas, or things you are wondering about to your list of noticings and wonderings.

Key Outcomes

- Get a better understanding of the problem by playing with a variety of ideas before solving the problem.
- Identify and pull together the most promising solution path from multiple representations or multiple strategies.
- Discover and note deeper, hidden relationships that emerge as you play with various possibilities.

III. *Playing with Clues*

Sample Activity: What If...

Format: students working in groups of three to five.

"Clues" is a useful shorthand for the longer phrase "quantities and relationships you noticed or wondered about."

One way to understand how a particular clue is a useful part of the problem is by changing it and noticing how the problem changes. I might change the value of a clue, or even pretend I don't know it. My goal is to focus on what changes in the problem, and how I can use that information to understand the clue better.

- 1) With your group, go through each clue or set of clues. Write down ways you could change the value of quantities in that clue or the constraint that it imposes.
- 2) For each clue, play out the problem a little bit with the new value(s). How would the problem change? Would it be easier or harder? What would be easier or harder about it? Would the results be different?
- 3) Now go through the clues again, this time ignoring one clue or set of clues at a time, pretending that information was never given.
- 4) How does the problem change when those clues are ignored? Does it make a simpler version you can use to learn more? Does it change the number of possible answers? What else changes?
- 5) After you've gone through all the clues, look back at the original problem. What new understanding have you gained? Do you see more uses for any of the clues? Do any of the clues seem more necessary (or unnecessary)?

Key Outcomes

- Explore how the problem was constructed.
- Generate additional information and perspective by changing or ignoring clues.
- Gain better understanding of the problem by thinking about simpler (and harder) versions.

IV. *Playing with Representations*

Sample Activity: How Else Can We Say It?

Format: students working in groups of three to five (depending on the number of clues in the problem).

Sometimes the clues are said one way, but if you said them (or wrote them or drew them) just a little differently, you would see different relationships. In this activity, try to express the clues as many different ways as you can.

- 1) Each person picks one clue from the problem and rewrites it or draws a picture of it or somehow changes *how* it's said without changing *what* is said.
- 2) After a few minutes, each person passes the clue to their left. Try to add another way to say (or draw or represent) the clue, adding to those already written down.
- 3) Keep passing clues to the left as long as you can come up with new ways to express them.
- 4) Once you've run out of ideas working individually, hold a group discussion:
 - a. Check if any of the different expressions changed *what* the clues really meant. In this activity you don't want to change what the clues mean. You just want to get a new perspective on what they mean.
 - b. Did any new information or perspective emerge that helps you see an approach to solving the problem?

Key Outcomes

- Understand the problem better using multiple representations of key information.
- See the problem from a fresh perspective.

Examples: The Bouncing Ball (AlgPoW)

The goal of these lessons is for the students to reflect on their own process in exploring the information given in a problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and instead focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "algpow-teachers" discussion group (<http://mathforum.org/kb/forum.jspa?forumID=528>) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

I. *Calculating (and Noticing) as You Go*

Student 1: I know $\frac{1}{3}$ of 27 is 9. So $\frac{2}{3}$ is 18.

Student 2: Right. So. For the next bounce do the same with 18. $\frac{1}{3}$ of 18 is 6 and $\frac{2}{3}$ is 12.

Student 1: We won't be able to do this forever because eventually 3 won't go into our new number evenly.

Student 2: Well, we could switch to inches. So far we're doing feet.

Student 1: Okay. We can do it for feet once more anyway. $\frac{1}{3}$ of 12 ft. is 4 ft., so $\frac{2}{3}$ is 8 ft. But for the next one 3 won't go evenly into 8.

Student 2: I guess it will if we switch to inches because 3 goes into 12 and we'll multiply by 12 to get inches.

Student 1: I wonder if there is a quicker way? Every time we divide by 3 and multiply by 2 to find out what the new number is.

Student 2: So, for the third bounce that would be like dividing by 27 and multiplying by 8. That works. $\frac{27}{27}$ is 1, and multiply by 8 to get 8ft.

Student 1: So the fourth bounce would be dividing by 81 and multiplying by 16. Yeah, but 81 is bigger than 27, so now's when we need to switch to inches.

Student 2: I guess we could also just leave it as a fraction of feet. 27 divided by 81 is $\frac{1}{3}$. Multiply that by 16 and we have $\frac{16}{3}$ ft. or 5 and $\frac{1}{3}$ ft.

Student 1: Excellent! That's the answer to the first question. So now I guess we need to write a formula for our idea of dividing by 3 and multiplying 2 the same number of times as bounces.

II. Playing with Strategies

Possible Strategies

- Tables and Patterns
- Simpler Problem
- Change the Representation

Results

- **Tables and Patterns (3 minutes):**

Bounce #	Starting Height (ft.)	Multiply by	Ending height (ft.)
1	27	$\frac{2}{3}$	18
2	18	$\frac{2}{3}$	12
3	12	$\frac{2}{3}$	8
4	8	$\frac{2}{3}$	5.33333333
5	5.33333333	$\frac{2}{3}$	3.55555556
6	3.55555556	$\frac{2}{3}$	2.37037037

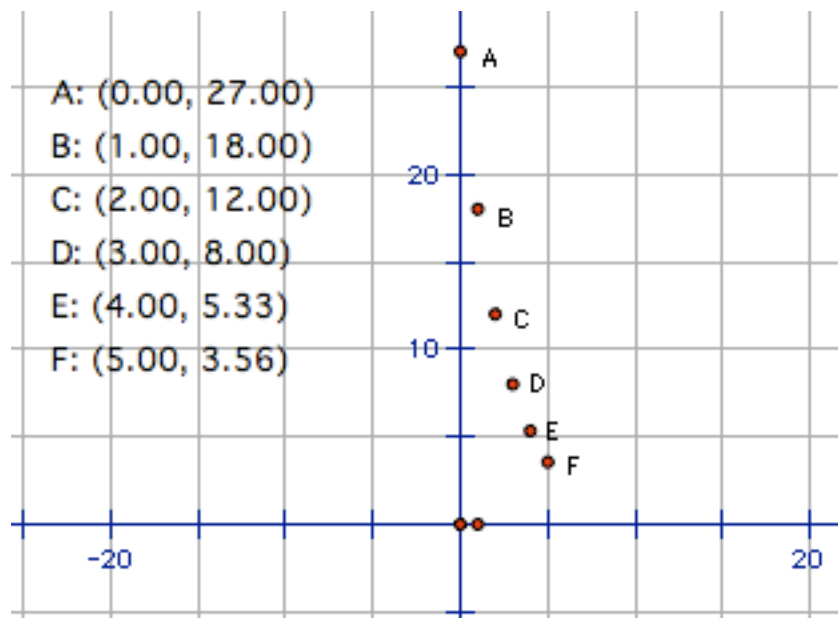
I used Excel. (*Time's up*)

- **Simpler Problem (3 minutes):**

Suppose it bounced back up to its full height each time. Each time you'd multiply by 1 and just get the same number. On the fourth bounce you would have multiplied by 1 four times: $27 * 1 * 1 * 1 * 1$ or 27. How would you write that as a formula? I guess with exponents: $27 * 1^n$. (*Time's up*)

- **Change the Representation**

I made a graph of the bounce and the height.



It doesn't look linear, or quadratic.

New Noticings and Wonderings

- *Tables and Patterns* using Excel seems like a quick way to get values calculated.
- We wondered if the formula would be more obvious from the Table if we didn't actually do the calculations, but left them written out: e.g. $27 * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3}$?
- *Simpler Problem* seemed to give us a formula, although we're not sure how to calculate quickly with it. It does seem like the exponential formula is where the *Tables and Patterns* idea above is going. $27 * \left(\frac{2}{3}\right)^n$
- We guess we can use the calculator to calculate the exponential expressions. Should we try to avoid having to do that?
- From *Changing the Representation* we can see that the shape of the graph means we need a different kind of formula than usual. We also notice the height changes by a lot at first, and then slows down. We think of exponential functions getting big quickly, but this is more like the half-life idea, where it gets small quickly.
- We wonder if we could **work backwards** using these ideas to figure out the extra?

III. Playing with Clues

Clues

The clues we found in the problem were:

- The ball starts 27 ft. high.
- When it bounces, it only goes back up 2/3 as high.

Changing Values

Suppose the ball started at 36 ft. and only bounced up halfway.

Bounce #	Height (ft.)
0	36
1	18
2	9
3	9/2
4	9/4
5	9/8

Noticing and Wondering:

- We thought 36 was a good number to use with 1/2, but after a couple of bounces we're into odd numbers. It would have been better to pick a power of 2, like 32 or 64.
- Hmm, speaking of powers, maybe we should use exponents in our solution somehow, if we don't want to go step-by-step.
- It still seems easier to deal with halves than two-thirds. You just keep multiplying by 1/2. When we multiply by a half, we don't mind having fractions for answers, for some reason.
- We guess we can just do the same thing with 2/3, multiply by it every time.

Ignoring Clues

We decided to ignore all the numbers given in the problem and just use variables to see if this helps us see what is going on.

H = original height

B = height fraction for each bounce

n = number of bounces

1st bounce – $H * B$

2nd bounce – $(H * B) * B$, which is $H * B^2$

3rd bounce – $(H * B^2) * B$, which is $H * B^3$

nth bounce - $H * B^n$

Noticing and Wondering

Not thinking about the numbers forced us to write out what was happening each time. We couldn't simplify it and that made the multiplication pattern more obvious.

Is there an easy way to calculate what you get when you plug in numbers for B^n , other than use the calculator?

For the extra, could we work backwards from $40 * B^{10} = 1$ to figure out what B is?

IV. Playing with Representations

Clue: "Each time it hits the floor, it bounces back up to a height 2/3 of the height from which it fell."

- We could say that each time it loses 1/3 of its height.

Does that make it harder or easier to solve? We think it makes it harder because you add a step where you first have to subtract what was lost, rather than just figure out the new height.

Height	Height lost	New height
27	$(27)1/3$	$(27)2/3$
$(27)2/3$	$(27)2/9$	$(27)4/9$
$(27)4/9$	$(27)4/27$	$(27)8/27$
$(27)8/27$	$(27)8/81$	$(27)16/81$

Our table did help us see that you find the new height by multiplying by 2/3 squared, or cubed, or to the 4th power, etc., depending on which bounce you are on.

- We pictured the heights of the bouncing ball as like fraction bars and each time we divided their thirds into thirds to make it easy to figure out what 2/3 would be for the next height. This also showed us we were working with exponential expressions. The divisions got too small for the last bounces.

