



Algebra PoW Packet

The Bouncing Ball

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Welcome!

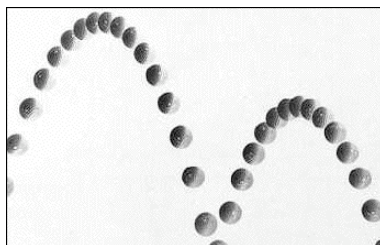
This packet contains a copy of the problem, the “answer check,” our solutions, teaching suggestions, a problem-specific scoring rubric, and some samples of the student work we received when *The Bouncing Ball* was first used in April 2005.

We invite you to visit the PoW discussion groups to explore these topics with colleagues. From the Teacher Office use the link to “PoW Members” or use this URL to go to *algpow-teachers* directly: <http://mathforum.org/kb/forum.jspa?forumID=528> [Log in using your PoW username/password.]

The Problem

We are revisiting *The Bouncing Ball*, Problem 3400 from the Library. It’s designed to explore the idea of exponential decay, and even students who are not yet familiar with the general exponential function should be able to come up with it as they work through the steps of the problem. It also includes an Extension question that could lead to some very nice conversation, which we’ll talk about in the Teaching Strategies section of this document. Here’s the problem:

The Bouncing Ball



A rubber ball is dropped from a height of 27 feet onto a concrete floor. Each time it hits the floor, it bounces back up to a height $\frac{2}{3}$ of the height from which it fell.

1. Calculate the height of each of the first four bounces.
2. Find a formula that gives the height of the n th bounce.
3. Use your formula to find the height of the 12th bounce to the nearest inch.

Extra: Suppose a different ball is dropped from 40 feet, and on its tenth bounce it reaches a height of 1 foot. What percent of the height from which it fell does that ball bounce back up each time it hits the floor?

Extension: Tell if you agree or disagree with the following statement and explain why. Support your opinion with any relevant mathematical ideas.

Since the original ball always bounces back up to $\frac{2}{3}$ of the height from which it fell, it can never actually stop bouncing. Even as the bounces get very small, there is always some height from which it falls and so it always bounces back up $\frac{2}{3}$ of that height. Thus, it never stops bouncing.

Answer Check

After students submit their solution, they can choose to “check” their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually **get** the answer) we provide hints and tips for those whose answer doesn’t agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation:

The heights of the first two bounces are 18 ft and 12 ft. Be sure you’ve found all four bounces. One possible formula for the n th bounce is...height = $27 \cdot (\frac{2}{3})^n$. Be sure that you’ve explained how you

came up with your formula and why it works. Also remember to use it to find the height of the 12th bounce to the nearest inch.

If your answers **don't** match ours:

- * did you multiply each bounce height by $\frac{2}{3}$ to find the next bounce height?
- * did you think about the idea that each bounce adds another "multiplying by $\frac{2}{3}$ " to the previous bounce calculation?
- * did you remember to convert your answer to question 3 from feet to inches, and round off carefully?
- * did you check your arithmetic?

If any of those ideas help you, revise your answer. You can also leave a comment that tells what you changed and why you changed it. If you're still stuck, leave a comment that tells where you're having trouble.

If your answers **do** match ours,

- * have you clearly shown and explained the work you did?
- * did you explain how you came up with your formula and why it works?
- * did you make any mistakes along the way? If so, how did you find and fix them?
- * are there any hints that you would give another student?
- * are you confident that you could solve another problem like this successfully?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did - you might answer one or more of the questions above.

Our Solutions

Question 1

I know that the ball falls from 27 feet and bounces back up $\frac{2}{3}$ of that height, so I multiplied 27 times $\frac{2}{3}$ and got 18 feet for the height of the first bounce. Then, since it went up 18 feet, it fell from 18 feet for the second bounce and bounced back up $\frac{2}{3}$ of 18 feet. I made a table showing the four bounces:

- Bounce 1: $\frac{2}{3} * 27 \text{ feet} = 18 \text{ feet}$
- Bounce 2: $\frac{2}{3} * 18 \text{ feet} = 12 \text{ feet}$
- Bounce 3: $\frac{2}{3} * 12 \text{ feet} = 8 \text{ feet}$
- Bounce 4: $\frac{2}{3} * 8 \text{ feet} = \frac{16}{3} \text{ feet, or } 5 \frac{1}{3} \text{ feet}$

The first four bounce heights were 18 feet, 12 feet, 8 feet, and $5 \frac{1}{3}$ feet.

Question 2

I noticed that each new bounce took the previous bounce height calculation and multiplied it by another $\frac{2}{3}$. I rewrote my chart from question 1 to show that pattern, and then wrote the number of times I multiplied by $\frac{2}{3}$ as an exponent:

- Bounce 1: $27 * \frac{2}{3} = 27 * \frac{2}{3}^1$
- Bounce 2: $27 * \frac{2}{3} * \frac{2}{3} = 27 * \frac{2}{3}^2$
- Bounce 3: $27 * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} = 27 * \frac{2}{3}^3$
- Bounce 4: $27 * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} * \frac{2}{3} = 27 * \frac{2}{3}^4$

I noticed that the number of times I multiplied the 27 by $\frac{2}{3}$ matches the number of the bounce, so the nth bounce should reach a height of $27 * \frac{2}{3}^n$ and I wrote the formula as:

$h = 27 * \frac{2}{3}^n$ where h is the height in feet and n is the bounce number.

Question 3

Using my formula from question 2, I calculated the height in feet of the 12th bounce by letting $n = 12$, and then I converted the result to inches:

$$27 * \left(\frac{2}{3}\right)^{12} = .208098 \text{ feet}$$

$$\frac{.208098 \text{ feet}}{1} * \frac{12 \text{ inches}}{1 \text{ foot}} = 2.497 \text{ inches}$$

Although it's almost 2 1/2 inches, which would round up to 3, it's just a little less than 2 1/2, and the 2.4 says to round down.

To the nearest inch, the 12th bounce will rise 2 inches.

Extra:

The ball drops from 40 feet, and reaches a height of 1 foot on the tenth bounce. I let p = the percent in decimal form of the falling height that it rebounds each time. Then I wrote a formula like the one from the problem but with a starting height of 40 instead of 27 and the unknown p instead of $2/3$, so I had:

$$40 * (p)^n = \text{height of } n\text{th bounce}$$

I knew that the 10th bounce had a height of 1 foot, so I substituted 10 for n and 1 for the height. Then I solved my equation by dividing both sides by 40 and then taking the tenth root of each side:

$$40 * (p)^{10} = 1$$

$$(p)^{10} = 1/40 \text{ or } 0.025$$

$$[(p)^{10}]^{0.1} = (0.025)^{0.1}$$

$$p = .6915$$

$$p \text{ as a percentage} = .6915 * 100 \text{ or approximately } 69\%$$

The ball bounces back up to 69% of the height from which it fell.

Extension:

Students should comment that while in the perfect theoretical math world the ball never stops moving, in reality the bounces become so imperceptible that it does in fact stop. Advanced kids might talk about limits at infinity or Zeno's Paradox, but as long as they get the general idea and say something sensible, commend them.

Teaching Suggestions

The math in this problem is fairly straightforward and doesn't really require a lot of formal algebra other than using variables in the exponential decay formula for questions 2 and 3. It's really designed to give kids a chance to explore the idea of exponential decay, and certainly that's easily linked in class to exponential growth as well. Having students plot the data points that they generate is a nice way to bring out the shape of the exponential decay function, dramatizing the large initial decreases and the smaller subsequent ones.

The Extra problem provides an opportunity to extend knowledge students may have about square roots, to see if they realize that higher roots exist as well and that the inverse of raising to the tenth is taking a tenth root. Students familiar with exponent properties might be able to see that the expression needs to be raised to the $1/10$ th power and be able to do that on their calculators/

The Extension is really there because it's a fun question to think about. On some levels it's more of a physical problem than a mathematical one, and we're pretty sure that most kids will think that the ball does in fact stop bouncing. It's a great opportunity to raise Zeno's Paradox, which kids are usually fascinated by, and even to introduce the idea of a limit as the function moves toward 0 as the number of bounces moves toward infinity.

Scoring Rubric

The **problem-specific rubric** is something we write for every problem for use by those who are assessing student work. It spells out what we expect from students in three areas of problem solving and three areas of communication. The goal is to assess a student response within each category as it relates to the specific criteria for that category. This approach allows you to retrieve more targeted information on the students' areas of strength and weakness.

In the most general sense, *Interpretation* usually includes understanding the given information, including any diagram, attempting to answer all parts of all of the questions asked, and exhibiting understanding of any fundamental math concepts necessary to solve the problem. *Strategy* is usually then applying all of that knowledge in a systematic and mathematically sound way that doesn't rely on any lucky guesses. *Accuracy* simply refers to executing your strategy correctly. Note that even an incorrect strategy can be done accurately. *Completeness* is showing and explaining the thinking and work you did to reach your answer, and *Clarity* is presenting that explanation in a way that is easy for someone else to read and understand. *Reflection* includes such things as thinking about the reasonableness of your answer, checking it, tying the problem to past problems, and summarizing the

key ideas in your solution.

A **generic student-friendly rubric** can be downloaded from the *Scoring Guide* link on any problem page. We encourage you to share it with your students to help them understand our criteria for good problem solving and communication.

The problem-specific rubric is shown on the next page, followed by sample student solutions from when *The Bouncing Ball* first appeared in April of 2005.

**Happy Problem
Solving!**

We hope these packets are useful in helping you make the most of the AlgPoWs. Please let me know if you have ideas for making them more useful.

– RIZ

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Algebra Problem of the Week Scoring Rubric for *The Bouncing Ball*

For each category, choose the level that *best describes* the student's work.

Problem Solving		Novice	Apprentice	Practitioner	Expert
Interpretation	shows understanding of only one or two of the concepts involved - see the Practitioner column	shows understanding of most but not all of the concepts in the Practitioner column (for example, can calculate bounce heights manually but doesn't see that a formula can be produced to make that repetitive calculation unneeded....) does not complete all parts of the problem	shows understanding that: each bounce height is $\frac{2}{3}$ of the previous one $\frac{2}{3}$ is multiplied repeatedly and that an exponent can be used to represent the repetitive multiplication writing a formula allows direct calculation of the height of any bounce without the previous height feet can be converted into inches	solves the main problem and the Extra correctly, and is at least a Practitioner in Strategy understands that a tenth root can be used to undo a tenth power and applies that in the Extra	
Strategy	has few ideas that will lead them toward a successful solution	picks an incorrect strategy, or relies on luck to get the right answer uses Guess and Check, either for entire solution or to solve an equation	is able to determine the parameters in the formula without guessing and by using algebraic reasoning might determine the relationship is exponential by noticing a pattern in repeated calculations, making a graph, making a table, or using prior knowledge	uses two separate strategies or an unusual or sophisticated strategy	
Accuracy	work contains many errors	work is mostly accurate, with a few errors uses incorrect units	work is accurate and contains no arithmetic mistakes uses appropriate units in questions 1 and 3 and rounds to nearest inch correctly in question 3	generally not possible - can't be more accurate than Practitioner	
Communication					
Completeness	has written very little that tells or shows how they found their answer	does not define variable(s) explains the steps used to find the answer but shows very few of the calculations and work OR shows the work but does not explain the thinking behind it	defines variable(s) explains all of the steps taken to solve the problem explains how/why they know that the relationship is exponential in nature, or why they believe it's something else	adds in useful extensions and further explanation of some of the ideas involved, such as taking a shot at the Extension question the additions are helpful, not just "I'll say more to get more credit"	
Clarity	explanation is very difficult to read and follow	explanation isn't entirely unclear, but would be hard for another student to follow explanation is long and is written entirely in one paragraph explanation contains many spelling and typing errors	explains things in such a way that another student would understand the work and thinking used in the solution makes an effort to check their formatting, spelling, and typing (a few errors are fine as long as they don't make it hard to read)	formats things exceptionally clearly answer is very readable and appealing	
Reflection	The items in the columns to the right are considered reflective. They could be in the solution or the comment left after viewing the Math Forum's answer. did nothing reflective	checked answer in some way (in addition to viewing the answer provided by the Math Forum) reflected on the reasonableness of their answer did one reflective thing	connected the problem to prior problems or experiences explained where they are stuck summarized the process they used did two reflective things	commented on and explained the ease or difficulty of the problem revised and improved their work	did three or more reflective things or did an exceptional job with two of them

Student Solutions
With focus on Interpretation

As noted in the rubric, interpretation for this problem involves understanding that each bounce height is $\frac{2}{3}$ of the height from which the ball fell, that there is repetitive multiplication involved, and that such repetitive multiplication can be modeled with exponents.

The sample student solutions included in this packet represent a broad range of both writing and problem solving skills. They also show a range of interpretation, and we've tried to address each student's individual misunderstanding or weakness with comments that suggest what might be a good next step for that student.

Amanda
age 13

Interpretation
Novice

i think that the bounce ball is getting higher and higher as it goes. well, the first ball goes to 4, 2nd goes to 6, 3rd goes to 8, etc.

Amanda has missed the whole idea that the bounce heights decrease. I'd start her with a rebound of $\frac{1}{2}$ rather than $\frac{2}{3}$ and start with a single bounce and then move to multiple bounces to help her understand the problem.

Blake
age 15

Interpretation
Novice

The height of the first four bounces are, 1st bounce= 25.5 ft. 2nd bounce= 24 ft. 3rd bounce= 22.5 ft 4th bounce= 21ft.

The height of the 9th bounce would be 13.5 ft, and the height of the 12th bounce would be 9 ft.

Figured out the decimal for $\frac{2}{3}$ which equals 1.5 and took 1.5 away from 27 ft 12 times and got my answers. 1st bounce 25.5 ft, 2nd bounce 24ft, 3rd bounce 22.5ft, 4th bounce 21 ft., 9th bounce 4.5 ft., 12th bounce 9ft.

Blake understands the repetitive nature of the problem but misunderstood the role of the $\frac{2}{3}$ and is subtracting from each bounce. He also did not try to answer the second question and produce a formula. As with Amanda, I'd start him with a simpler version of the problem.

Sarah
age 15

Interpretation
Apprentice

1) The first bounce bounced back up to 18 feet, the second to 12 feet, the third to 8 feet and the fourth to 5 feet high.

2) The formula to find the height of bounce (X) depending on the number of bounces (N) is $X = 0.6$ repeated multiplied by N.

3) The height of the twelfth bounce is equal to 1 foot, 6 inches.

1) [1st bounce] $27(0.6\dots) = 18\text{ft.}$
[2nd bounce] $18(0.6\dots) = 12\text{ft.}$
[3rd bounce] $12(0.6\dots) = 8\text{ft.}$
[4th bounce] $8(0.6\dots) = 5\text{ft.}$

2) let X= the height of the bounce
let N= the number of bounces
 $X = 27(0.6/N)$

3) $X = 27(0.6/12)$
 $X = 27(0.05\dots)$
 $X = 1.5\text{ ft.}$
 $X = 1'6"$

Sarah understands that the problem involves repeated multiplications to find the bounce heights, but seems to see it as more of an iterative process and does not seem to understand that she can use an exponent to create the repeated multiplication, switching instead to division in her formula. I'd ask her to think about writing expressions for her first four answers that all start with 27 to see if that helps her see the exponential nature of the problem.

Sangil
age 14

Interpretation
Apprentice

first start at 27 feet after one bounce , it bounce off 2/3 of first.

1. first -> 27 feet
second -> $27 \cdot \frac{2}{3} = 18$ feet
third -> $18 \cdot \frac{2}{3} = 12$ feet
fourth -> $12 \cdot \frac{2}{3} = 8$ feet.
2. first $27 \cdot \frac{2}{3}^0$
second $27 \cdot \frac{2}{3}^1$
third $27 \cdot \frac{2}{3}^2$
fourth $27 \cdot \frac{2}{3}^3$
fifth $27 \cdot \frac{2}{3}^4$
nth $27 \cdot \frac{2}{3}^{(n-1)}$
3. after 12 times it would be $27 \cdot \frac{2}{3}^{11}$
 $= 0.31214\dots$
 $= 0.31214$ feet

Sangil has understood a lot of the ideas in the problem but missed the idea that it starts with the ball dropping from 27 feet and instead has it bouncing up to 27 feet on the first bounce. Sangil also missed the idea that the answer to #3 should be expressed in inches. I'd ask Sangil to think carefully about the first bounce again but commend his work overall. Note that Sangil would be a Practitioner in Strategy because he has executed his understanding of the problem perfectly.

Margaret
age 12

Interpretation
Practitioner

1) At first, I used a table of values to figure out the first four bounces by multiplying by 2/3 every bounce:

if n = number of bounces, h = the height of the ball

n		1		2		3		4	

h		18		12		8		16/3	

2) I decided to look at the table of values to figure out the equation. I realized that this is what I did in my head:

n		1		2		3	

h		$27 \cdot \frac{2}{3}$		$27 \cdot \frac{2}{3} \cdot \frac{2}{3}$		$27 \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$	

With this, I saw the pattern to create the equatin: $27(\frac{2}{3})^n$

3) In this scenario, $n = 12$. Moreover, the equation requires the height to be in inches, so I converted 27 feet to inches:

$$\begin{aligned} 27 (12) &= 324 \\ 324 (\frac{2}{3})^n & \\ 324 (\frac{2}{3})^{12} & \\ 324 (4096/531441) &= 2.49718\dots \end{aligned}$$

I rounded this to 2 inches.

Margaret has clearly understood the key ideas in the problem. She would also score well in Clarity because she has organized her work and presented it very clearly. I might ask Margaret to complete her formula/equation for question 2 by having two sides to it, but overall I'd commend her for her fine work and ask her if she would like to try the Extra and the Extension.

Daisuke

age 13

Interpretation

Expert

1. Since the ball bounced $\frac{2}{3}$ the height from which it fell,

First Bounce: $27 * \frac{2}{3} = 18$ 18 feet
Second Bounce: $18 * \frac{2}{3} = 12$ 12 feet
Third Bounce: $12 * \frac{2}{3} = 8$ 8 feet
Fourth Bounce: $8 * \frac{2}{3} = \frac{16}{3}$ $\frac{16}{3}$ feet

2. 27 feet is the starting height at which the ball is dropped, and each time the ball hits the floor, it bounces back up two thirds the previous height from which it falls.

So, the first bounce can be expressed as $27 * \frac{2}{3}$, or $27 * (\frac{2}{3})^1$,
The second bounce $27 * \frac{2}{3} * \frac{2}{3}$, or $27 * (\frac{2}{3})^2$,
The third bounce $27 * \frac{2}{3} * \frac{2}{3} * \frac{2}{3}$, or $27 * (\frac{2}{3})^3$, and so on.
Therefore, you get the formula $27 * (\frac{2}{3})^n$ to find the height in feet of the nth bounce.

3. Substituting 12 as n,

$$27 * (\frac{2}{3})^{12}$$

Since 1 foot equals 12 inches, in order to find out the height to the nearest inch, you multiply the whole equation by 12.

$$\begin{aligned} & 27 * (\frac{2}{3})^{12} * 12 \\ &= 27 * 12 * (\frac{2}{3})^{12} \\ &= (3^3) * (3 * 2^2) * \frac{[(2^{12}) / (3^{12})]}{1} \\ &= 3^4 * 2^2 * \frac{[(2^{12}) / (3^{12})]}{1} \\ &= \frac{[2^{12} * 12]}{[3^{12-4}]} \\ &= \frac{2^{14} * 3}{3^8} \\ &= 2.497... \\ &= \text{about 2 inches} \end{aligned}$$

So, on the 12th bounce, the height will be about 2 inches.

Extra:

Putting $27 * \frac{2}{3}^n$ into a more multiuse or general formula, you can express it as

$$h = s * b^n,$$

where s represents the starting height at which the ball falls from, b represents the percentage (in decimal form) or fraction of how much the ball bounces back up after it hits the floor, n represents the nth bounce, and h represents the height of the nth bounce.

Substituting 40 as s, 1 as h, and 10 as n,

$$1 = 40 * b^{10}$$

Simplifying the equation,

$$\begin{aligned} b^{10} &= \frac{1}{40} \\ b &= (\frac{1}{40})^{(1/10)} \\ b &= 0.6915... \\ b &= \text{about 0.69} \end{aligned}$$

Therefore, the ball bounces back up about 69% each time it hits the floor.

Extension:

In this case, the ball will keep in bouncing, because it is ignoring friction and air resistance. However, in real life, it will stop bouncing after a while. This is because of several different reasons. Firstly, because of air resistance, it will stop bouncing after a while, and the ball will lose some of its energy bit by bit, until it is completely gone. Secondly, because when the ball bounces, the atoms in the ball rub against each other, it loses some more energy in the form of heat. Friction may also play a part in why the ball stops as well.

Daisuke has done an excellent job of not only solving the main problem but also of applying his understanding to the Extra question, including thinking about how to solve an equation involving a variable raised to the tenth power. He's also presented his work clearly and completely, earning high scores in Completeness and Clarity. Given Daisuke's clear math skill and excellent solution, I'd ask him to think about the Extension. He's made some great observations about the physics of a bouncing ball, so I'd want to try to introduce him to the idea of a limit. I'd ask him what happens to the height of the bounce after 20 bounces, then after 30, after 100, and after infinite bounces. I'd hope that by evaluating those questions he might begin to think about the result growing infinitely small and that we might have an interesting conversation about that topic.