Geometry PoW Packet

Leaning Ladder

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Welcome!

This packet contains a new problem, the "answer check," our solution and scoring rubric, a note about possible common mistakes we may see, and ideas for implementing the problem in the classroom.

You can access the discussion group via the link to "PoW Members" in your Teacher Office, or use this URL to go to geopow-teachers directly: http://mathforum.org/kb/forum.jspa?forumID=529 [Login to the discussions using your PoW username/password.] Let us know if you see anything interesting when using this problem in your classroom.

The Problem
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The most compelling aspect of this problem from my perspective was that it didn't work exactly the way I thought it would—see my lament at the end of the solution later in this document. Unless your students are algebra studs (and they certainly needn't be!), they will have a tough time finding the exact spot when the conditions are satisfied. Fortunately, the problem doesn't ask them to do that. They simply have to figure out whether or not the conditions *can* be satisfied. To get them started, you might simply present the scenario, maybe using some sort of manipulative, or a *Sketchpad* sketch, and ask them what they think happens as the ladder moves. How are the distance of the base from the wall and the height of the top up the wall related at any given time? If you move the base in, does the height move up the same amount? I'm guessing that some students might be surprised to see that it doesn't.

The problem relies on the idea of continuity—if something is moving smoothly and is too big at the beginning and too small at the end, it must be just right somewhere in the middle. Again, there's no numerical answer necessary, though students will need to provide some evidence for their conclusion. It may even be the case that there can be some good conversation in the classroom between the two camps.

Answer Check
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After students submit their solution, they can choose to "check" their answer by looking at the answer that we provide. Along with the answer itself (which never explains *how* to actually get the answer), we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

If your students do submit and see that their answer is wrong, please encourage them to keep working on the problem. We get a lot of submitters who get the problem wrong and look at our answer (so they know they're wrong), and then never do anything else. It may be that they just figure they're wrong, and that's the end of it. But the real learning happens not in solving the problem correctly, but in revising and thinking about the problem more once you've solved it incorrectly the first (or second, or third!) time. Those students learn far more than the students who get it right first try. We've actually done research about this, tracking students doing the PoWs over the course of a year and measuring the gains they make in terms of their ability to connect problems and concepts, their autonomy, and their strategy use. So encourage them to revise, even if it's only on paper!

Our Solution
page 2

We have only made our solutions available to mentors in the past, as we felt using authentic student work provided better examples of how they actually solve the problem. However, we also realized that sometimes having the solution ahead of time could be helpful, as we often include tips for how to support students in different areas of their work, or in thinking about different parts of the problem. When appropriate, we also include multiple ways to solve the problem.

Scoring Rubric
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The problem-specific rubric is something we write for every problem for use by those who are assessing student work. It spells out what we expect from students in three areas of problem solving and three areas of communication. It is designed to give students targeted information on their strengths and weaknesses. It's not intended to produce a "grade", but rather help students begin to think about where they're strong and where they could improve. You might share the problem-specific rubric with your students, or you might prefer to give them the generic rubric that's available from the Teacher Documents page at http://mathforum.org/pow/teacher/.

Common Mistakes right here!

We haven't used this problem with students, so we can't say for sure what might happen. My guess is that some students will think that the base, or maybe the height, must start and end a whole number of feet from the wall. Others will think that because they can't actually find a place where it works, it can't work.

Good luck!

Good luck to you and your students. We hope to get feedback and ideas from you on the geopow-teachers discussion group starting April 7, 2008.

—Annie

Problem

Leaning Ladder

A 15 foot ladder is leaning against a wall. As the base of the ladder is moved closer to the wall, the top of the ladder slides up the wall.

Are there any conditions under which moving the base one foot closer to the wall would result in the top of the ladder also sliding up the wall one foot? How do you know?

Answer Check

Yes, there are conditions under which moving the base one foot closer to the wall would result in the top of the ladder also sliding up the wall one foot.

If your answer **does not** match our answer,

- did you notice that the ladder forms a right triangle with the wall and the floor?
- note that the ladder doesn't have to start a whole number of feet from the wall.
- remember that you don't actually have to find when it happens, just explain how you know that it does.

If any of those ideas help you, you might revise your answer, and then leave a comment that tells us what you did. If you're still stuck, leave a comment that tells us where you think you need help.

If your answer **does** match ours,

- have you given evidence to support your conclusion?
- did you make any mistakes or have any "aha!" moments along the way?
- are there any hints that you would give another student?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did—you might answer one or more of the questions above.

Our Solution

The key concept in this problem is the Pythagorean theorem.

Method 1: Guess and Check

Let's say that the base is 9 feet from the wall. By the Pythagorean theorem, the ladder now forms a 9-12-15 triangle, so the top is 12 feet up the wall. Move the base so that it's now 8 feet from the wall, and see if the top ends up at 13 feet.

$$8^2 + x^2 = 15^2$$

$$
\begin{aligned}
64 + x^2 &= 225 \\
x^2 &= 161 \\
x &= 12.67 \text{ feet}
\end{aligned}
$$

It didn’t end up at 13. In fact, it didn’t move far enough. If we reverse the situation, the base starts at 12 and the height is 9. Move the base to 11 and see if the height is 10.

$$
\begin{aligned}
11^2 + x^2 &= 15^2 \\
x^2 &= 104 \\
x &= 10.12 \text{ feet}
\end{aligned}
$$

Now it’s moved more than a foot. It stands to reason that somewhere in between those two starting positions for the base (9 feet and 12 feet), there is a spot that will work. The problem doesn’t ask us to actually find the place, just to explain whether or not such a place exists. I argue that it does - if it’s too little at first, and then becomes too much, somewhere in the middle it must be just right.

Method 2: Direct Calculation

To see if there is a place where this will work, we can try to calculate when that would be and see if we get a solution.

We have two right triangles. The first has a base of x, a hypotenuse of 15, and, by the Pythagorean theorem, a height of $\sqrt{(15^2 - x^2)}$. The second has a base of x - 1, a hypotenuse of 15, and a height of $\sqrt{(15^2 - (x - 1)^2)}$.

We want to calculate when the first height plus 1 is equal to the second height. So we do a lot of nasty algebra.

$$
\begin{aligned}
\sqrt{(15^2 - x^2)} + 1 &= \sqrt{(15^2 - (x - 1)^2)} \\
15^2 - x^2 + 2\sqrt{(15^2 - x^2)} + 1 &= 15^2 - (x - 1)^2 \\
- x^2 + 2\sqrt{(15^2 - x^2)} + 1 &= - (x^2 - 2x + 1) \\
- x^2 + 2\sqrt{(15^2 - x^2)} + 1 &= - x^2 + 2x - 1 \\
2\sqrt{(15^2 - x^2)} &= 2x - 2 \\
\sqrt{(15^2 - x^2)} &= x - 1 \\
225 - x^2 &= x^2 - 2x + 1 \\
0 &= 2x^2 - 2x - 224 \\
0 &= x^2 - x - 112
\end{aligned}
$$

This doesn’t look factorable, so I’ll use the quadratic formula to solve it.

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$x = \frac{1 \pm \sqrt{[1 - 4(1)(-112)]}}{2}$$

$$x = \frac{1 \pm \sqrt{(449)}}{2}$$

Taking the negative square root will result in a negative number, and since we’re talking about a length here, it must be positive. So we’ll only calculate the one with the positive square root.

$$x = \frac{1 + \sqrt{(449)}}{2}$$

$$x = \frac{1 + 21.19}{2}$$

$$x = 11.09481 \text{ feet}$$

The condition under which it will be true is when the base starts at 11.09481 feet (or so – the first expression for x gives the exact answer). That’s pretty hard to measure so accurately, so you couldn’t actually do it. But if you started with the base at 11 and moved it to 10, it would probably seem like the top would end up moving just about a foot.

Method 3: Exhaustive Spreadsheet

One way I explored the problem was to make a spreadsheet and see where things go from "too much" to "too little". I set the starting distance of the base from the wall to be integers from 15 to 0, then calculated the following:

- the resulting height (using the Pythagorean theorem)
- the base – 1
- the resulting height when base – 1
- the difference in the two heights

starting base	starting height	base - 1	height if base -1	height difference
15.00	0.00	14.00	5.39	5.39
14.00	5.39	13.00	7.48	2.10
13.00	7.48	12.00	9.00	1.52
12.00	9.00	11.00	10.20	1.20
11.00	10.20	10.00	11.18	0.98
10.00	11.18	9.00	12.00	0.82
9.00	12.00	8.00	12.69	0.69
8.00	12.69	7.00	13.27	0.58
7.00	13.27	6.00	13.75	0.48
6.00	13.75	5.00	14.14	0.39
5.00	14.14	4.00	14.46	0.31
4.00	14.46	3.00	14.70	0.24
3.00	14.70	2.00	14.87	0.17
2.00	14.87	1.00	14.97	0.10
1.00	14.97	0.00	15.00	0.03
0.00	15.00	-1.00	14.97	-0.03

Spreadsheets are cool! I saw that the height difference went from 1.2 feet to 0.98 feet when the starting base moved from 12 feet to 11 feet. See the top picture on the right. Then I extended the table, but this time used 0.1 foot increments between 12 and 11 for the starting base. The results are in the second picture.

starting base	starting height	base - 1	height if base -1	height difference
12.00	9.00	11.00	10.20	1.20
11.90	9.13	10.90	10.30	1.17
11.80	9.26	10.80	10.41	1.15
11.70	9.39	10.70	10.51	1.13
11.60	9.51	10.60	10.61	1.10
11.50	9.63	10.50	10.71	1.08
11.40	9.75	10.40	10.81	1.06
11.30	9.86	10.30	10.90	1.04
11.20	9.98	10.20	11.00	1.02
11.10	10.09	10.10	11.09	1.00
11.00	10.20	10.00	11.18	0.98

I kept going with the spreadsheet, using smaller and smaller intervals until I figured that the zero point was somewhere between a starting base of 11.0949 feet and 11.0948 feet, which struck me as close enough.

One distressing thing (for me, anyway) that happened when solving this is that an assumption I made turned out not to be true. When we first solved this problem as part of one of our "math" meetings at the Math Forum, I proposed the following theory: When the ladder forms a 45-45-90 triangle, the base will be 15/sqrt2 feet from the wall. Move the base 6 inches further from the wall than that, and call that the starting spot. Then move it 1 foot closer (which is 6 inches on the other side of isosceles). That's the solution!

Everyone agreed that it sounded plausible and we carried on. Turns out, however, that it doesn't work that way. It sure seems like it should (or at least it did to me!), but try it out. You'll find that it's close, but it's not exact. I'll leave it to you to think about why that is. I'm still annoyed by it, feeling like symmetry let me down somehow, but in the end, it does make sense that it wouldn't work. That doesn't mean I'm happy about it, though!

Geometry Problem of the Week Scoring Rubric for *Leaning Ladder*

For each category, choose the level that *best describes* the student's work

	Novice	Apprentice	Practitioner	Expert
Problem Solving				
Interpretation	doesn't seem to understand the problem	doesn't understand that there is a right triangle doesn't understand what is happening when the ladder moves	understands that the ladder forms a right triangle with the floor and the wall understands what happens when the base of the ladder moves attempts to answer the question	since there is no Extra, there is no way to be an Expert in Interpretation
Strategy	has no ideas that will lead them toward a successful solution	has a strategy that somehow relies on luck tries only whole numbers of feet for the distance that the base is from the wall without understanding the idea of continuity	has a strategy that relies on sound reasoning, not luck calculates how far the height moves for several different starting base distances and invokes (though probably doesn't use the exact language of) the idea of continuity to conclude that if there are slides that are too low and too high, there must be one between them that is correct	finds the exact instance when the conditions are met using algebraic techniques or finds a very good approximation using a spreadsheet
Accuracy	has made many errors	has made several mistakes or misstatements, or has used vocabulary incorrectly	makes no mistakes of consequence and uses largely correct vocabulary and notation	[generally not possible]
Communication				
Completeness	has written almost nothing that tells you how they found their answer	shows work without explanation, or gives an explanation without showing any work gives results without showing calculations	shows and explains the steps taken and why they are reasonable steps, which might include: • how related bases and heights are calculated • what any numbers resulting from calculations represent • why the next "guess" or position for the base was chosen	includes additional helpful information, doesn't just add more for the sake of adding more
Clarity	explanation lacks clarity and organization	explanation is difficult to follow length warrants separation into more paragraphs lots of spelling errors/typos	explains the steps that they *do* explain so that another student would understand (needn't be complete to be clear) makes an effort to check their formatting, spelling, and typing (a few errors are okay)	answer is clearly written and well-organized formats things exceptionally clearly
Reflection	*The items to the right are considered reflective, and could be in the solution or their comment:*	checks their answer (not the same as viewing our "answer check") reflects on the reasonableness of their answer	connects the problem to prior knowledge or experience explains where they're stuck summarizes the process they used	comments on and explains the ease or difficulty of the problem **revising their answer and improving anything**
	does nothing reflective	does one reflective thing	does two reflective things	does three or more reflective things or great job with two