Geometry PoW Packet

The Perimeter of an Octagon

March 24, 2008 • http://mathforum.org/geopow/

Welcome!

We hope that you are finding the recent packets useful and informative. We would really appreciate hearing from you about them. In particular, what has been most useful to you?

You can access the discussion group via the link to "PoW Members" in your Teacher Office, or use this URL to go to geopow-teachers directly: http://mathforum.org/kb/forum.jspa?forumID=529 [Login to the discussions using your PoW username/password.]

This packet contains a problem from the Library, the "answer check," our solution and scoring rubric, a note about possible common mistakes we may see, ideas for implementing the problem in the classroom, and some student solutions.

The Problem

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This week's problem is *The Perimeter of an Octagon*, Problem 3147 from the Library. This problem is actually modeled on a problem included on the 2003 Texas Assessment of Knowledge and Skills. In that version, the center point wasn't labled the center, and students were supposed to use the Pythagorean theorem to work towards the perimeter. But the way the picture was drawn made it appear that the point was in fact the center, which might have been confusing to some students. This led the Texas Education Agency to make a rather bizarre claim – that there are two possible correct answers for this problem. (The official press release is still available as of March 2008.)

In the press release, they said, "However, if a student used one of the 45-degree angles at the center of the octagon and trigonometry to solve the problem, he or she may have chosen answer H, which was 27 cm, or determined that there was no correct answer." But if you solve our version of the problem, you'll see that if that point really is the center, the rest of the drawing disagrees with itself!

We had a lot of fun talking about this problem at the Math Forum back in 2003. We especially like the idea of showing that a particular situation can't be true. It's not something that most students get to practice very often, yet it's a really important skill, and provides a nice twist on more traditional tasks.

This problem is a good opportunity to practice applying right triangle trigonometry relationships. At least one trig ratio will be necessary to answer the first question. Depending on what students assume to be "not true", a different ratio might be necessary to answer the second question.

You might choose to introduce the problem by putting the picture on the board and asking students what they notice about it. What seems to be true? What else might be figured out? What has the class studied that might prove useful in this situation?

For each problem, we will pick one category from the scoring rubric (see below) on which we'll focus. For *The Perimeter of an Octagon*, we're choosing "Interpretation", which basically means whether or not the student has understood the information in the problem and the question(s) posed, and whether or not they've attempted to answer all of the questions. In this problem, it might include understanding the goal of the problem (the idea that something is wrong and you have to fix it), knowing the properties of a regular octagon, pointing out an inconsistency in the given iformation, and attempting to find the resulting perimeter.

Answer Check

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After students submit their solution, they can choose to "check" their answer by looking at the answer that we provide. Along with the answer itself (which never explains *how* to actually get the answer, and in this case only gives two of the three answers), we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

Our Solution

page 3

We have only made our solutions available to mentors in the past, as we felt using authentic student work provided better examples of how they actually solve the problem. However, we also realized that sometimes having the solution ahead of time could be helpful, as we often include tips for how to support students in different areas of their work, or in thinking about different parts of the problem. When appropriate, we also include multiple ways to solve the problem.

Scoring Rubric
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The problem-specific rubric is something we write for every problem for use by those who are assessing student work. It spells out what we expect from students in three areas of problem solving and three areas of communication. The goal is to assess a student response within each category as it relates to the specific criteria for that category. This approach to assessing student work allows you to retrieve more targeted information on the students' areas of strength and weakness.

Sample Solutions
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As always, the sample student work included in this packet represents a broad range of both writing and problem solving skills. They also show a range of understandings, and we've tried to address each student's individual misunderstanding or weakness with comments that suggest what might be a good next step for that student.

For those who have scored Novice or Apprentice in Interpretation, we've included suggestions or questions that might help them improve their understanding or notice things they missed the first time. You'll see that students' "interpretation" of this problem runs the gamut from thinking there isn't enough information, to understanding the idea of the problem but not knowing enough about octagons or making faulty assumptions, to understanding the problem but not finding the resulting perimeter of the octagon, to coming up with a successful solution to the problem, to solving the Extra question as well. Phew!

For those who are a Practitioner in Interpretation, we've focused on elements of their communication that could be improved.

Common Mistakes
right here!

Some students assumed that there wasn't enough information to solve the problem, since the edgelength of the octagon isn't given. You might ask them why they think so and possibly point out the triangle formed in the picture. Others made assumptions about octagons that aren't true, including the idea that they're made from 30-60-90 triangles, or 45-45-90 triangles, or that the radii are equal to the edgelengths. Still others assumed that the apothem is equal to the edgelength. A few others measured the picture.

It's certainly a common mistake for students who have recently learned about 45-45-90 and 30-60-90 triangles to use them for every subsequent triangle! The same thing often happens when kids first learn the Pythagorean theorem – every triangle is a right triangle.

In many of these cases, you might simply ask them how they know their statement is true. If they suggest something along the lines of, "I just know it is", ask a question that causes them to come up with an impossible situation. For example, for students who say that the right triangles in the picture is a 30-60-90 triangle, you could ask them to draw all the right triangles in the picture and figure out the angle at the center of the octagon, given the 30 degree angle of their triangle. Alternatively, ask them to review the ratios of the sides of a 30-60-90 triangle and see if the two sides of the triangle given in the problem follow those ratios.

Good luck!

We're excited about providing these new resources to you. We hope to get feedback and ideas from you on the geopow-teachers discussion group starting March 24, 2008.

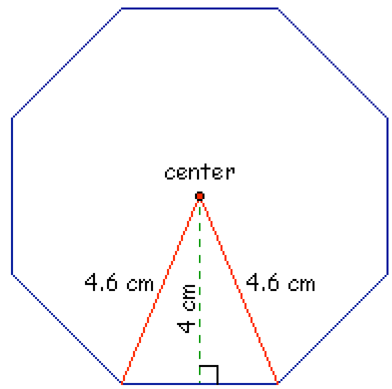
—Annie

Problem

The Perimeter of an Octagon

Given the regular octagon below, answer the following questions:

1. What's wrong with this picture?
2. If you fix what's wrong, what's the perimeter of the octagon?



Extra: Assume that the thing you found to be wrong is actually right. What else could you change to make things right? What's the resulting perimeter of the octagon?

Answer Check

There are three possible perimeters, depending on what you assume to be "false":

- 28.17 cm
- 26.51 cm
- 36.34 cm

If your answer does not match our answer,

- did you find the angle formed by the segments labeled 4 and 4.6 to be 22.5 degrees?
- you might want to read this Dr. Math thread about right triangles and angles – [http://mathforum.org/library/drmath/view/53939.html].
- remember that you can only assume that one thing is wrong at a time - the others must stay true.

If any of those ideas help you, you might revise your answer, and then leave a comment that tells us what you did. If you're still stuck, leave a comment that tells us where you think you need help.

If your answer does match ours,

- have you carefully explained what exactly is true and what's false in your situation?
- did you explain exactly how you found the perimeter?
- are there any hints you would give another student?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did—you might answer one or more of the questions above.

Our Solution

The key ideas in this problem are knowing some right triangle trigonometry and being able to figure out what's "wrong", which I'm guessing will be hard for some kids.

Let's look at what we're given. Since we're told that it's a regular octagon, we can figure out the central angle and the measures of the interior angles. With 8 sides, we find the measure of the central angle by doing

$$\text{central angle} = \frac{360}{8} = 45 \text{ degrees}$$

We can find the interior angles using the formula 180(n - 2)/n, where n is the number of edges.

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180(n - 2)
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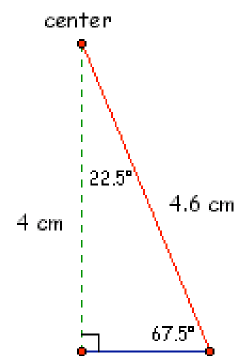
$$\text{interior angle} = \frac{}{n}$$

$$= \frac{180 * 6}{8}$$

$$= 135 \text{ degrees}$$

Let's look at the small right triangle formed in the picture. Since this is a regular octagon, the red line bisects the interior angle and the altitude (or apothem) bisects the central angle. This gives us the picture at the right.



So what's wrong with it? The fact that we're told something is "wrong" means that some of the information doesn't follow from the rest of the information. Given that it's a right triangle, we might have two thoughts. The first is to "check" using the Pythagorean theorem. But we're only given two sides of the triangle, so that won't help. The other thing we know about right triangles is right triangle trigonometry. (If we don't know it, there's a link in the answer check that might help them – feel free to share if they seem stuck: http://mathforum.org/library/drmath/view/53939.html.)

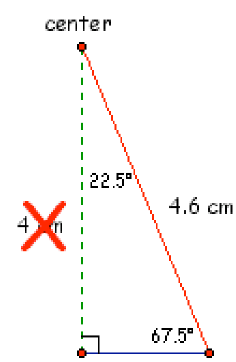
Let's consider the cosine of the 22.5 degree angle. Cosine is adjacent/hypotenuse, so we can set up the following equation and see if it's true:

$$\cos(22.5) \overset{?}{=} \frac{4}{4.6}$$

$$0.924 = 0.87$$

nope!

So something isn't right – either the 4, the 4.6, or the 22.5. Let's first assume that the 4 is incorrect.



We can find the correct length for the 4 by again using cosine.

$$\cos(22.5) = \frac{x}{4.6}$$

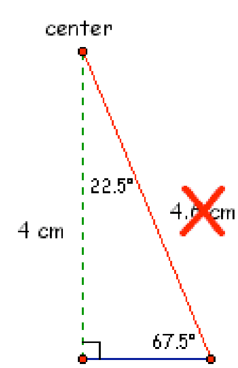
$$4.25 = x$$

Now we know the correct length for the apothem, assuming that the 4.6 and the 22.5 are right. Now we simply need to find the perimeter, which requires finding the length of the other leg of the right triangle. We could do by using the Pythagorean theorem on our right triangle, or by using sine and the 22.5. Sine uses the opposite leg and the hypotenuse.

$$\sin(22.5) = \frac{z}{4.6}$$

$$1.76 = z$$

Since this is half of a side of the octagon, we multiply this by 16 to get the perimeter. 1.76 * 16 = 28.16 centimeters. (They might get 28.17 if they don't round the value of x to 1.76, but instead use the entire number their calculator gives for the previous calculation. That's fine.)



So there's one way. The other is to assume that the 4 is correct, as is the 22.5, and the 4.6 is not.

Here we use much the same process, using cosine to find the correct length of the hypotenuse of the right triangle (p), and then using tangent (opposite over adjacent) to find the length of the other leg (q). The perimeter will again be 16 times that other leg.


```
              4
cos(22.5) = ---
              p

        p = 4.33

              q
tan(22.5) = ---
              4

     1.67 = q

perimeter = q * 16 = 26.51
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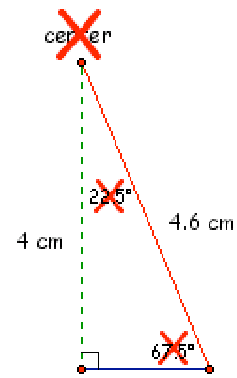

Those two methods will be the most likely that you'll see. But you've noticed that I keep mentioning the 22.5 as a possibility at well. How could that not be the top angle of that right triangle? If the point labeled "center" isn't really the center! Then the 4.6 and the 4 could be right, and we can simply use the Pythagorean theorem to find the length of the other leg (t).

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4.6^2 = 4^2 + t^2
21.16 = 16 + t^2
 5.16 = t^2
 2.27 = t

perimeter = 16 * t = 36.34
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That's perfectly valid as well. Another option would be to say that the octagon isn't regular, but then it's hard to fix what's wrong in a reliable way, so I'm advocating these three possibilities.

If they simply do the Pythagorean theorem and get a perimeter of 36.35 cm, they may assume that they're right, since that's one of the answers that we provide. But they need to have addressed the idea that "the center isn't the center" in order for their answer to be valid.

Geometry Problem of the Week Scoring Rubric for *The Perimeter of an Octagon*

For each category, choose the level that *best describes* the student's work

	Novice	Apprentice	Practitioner	Expert
Problem Solving				
Interpretation	doesn't seem to understand the goal of the problem thinks there isn't enough information	doesn't point out an inconsistency (might just use the Pythagorean theorem and find the perimeter) points out an inconsistency, but doesn't find the resulting perimeter makes false statements about regular octagons	seems to understand the goal of the problem knows the properties of a regular octagon points out an inconsistency in the given information attempts to find the resulting perimeter	is at least a Practitioner in Strategy and has solved the Extra correctly
Strategy	has no ideas that will lead them toward a successful solution	has a strategy that somehow relies on luck	has a strategy that relies on sound reasoning, not luck	considers and fixes all three possible inconsistencies and finds the correct perimeter in each case
Accuracy	has made many errors	has made several mistakes or misstatements, or has used vocabulary incorrectly uses trig ratios inappropriately	makes no mistakes of consequence and uses largely correct vocabulary and notation uses trig ratios appropriately	[generally not possible]
Communication				
Completeness	has written almost nothing that tells you how they found their answer	shows work without explanation, or gives an explanation without showing any work gives results without showing calculations	shows and explains the steps taken and why they are reasonable steps, which might include: • how any angles were calculated • what trig ratios were used how and why • how the perimeter was calculated	includes additional helpful information, doesn't just add more for the sake of adding more
Clarity	explanation lacks clarity and organization	explanation is difficult to follow length warrants separation into more paragraphs lots of spelling errors/typos	explains the steps that they *do* explain so that another student would understand (needn't be complete to be clear) makes an effort to check their formatting, spelling, and typing (a few errors are okay)	answer is clearly written and well-organized formats things exceptionally clearly
Reflection	*The items to the right are considered reflective, and could be in the solution or their comment:*	checks their answer (not the same as viewing our "answer check") reflects on the reasonableness of their answer	connects the problem to prior knowledge or experience explains where they're stuck summarizes the process they used	comments on and explains the ease or difficulty of the problem **revising their answer and improving anything**
	does nothing reflective	does one reflective thing	does two reflective things	does three or more things or great job with two

Lauren
age 15

Interpretation
Novice

You can't solve the above problem.

You can't find the perimeter of the octagon because the picture doesn't tell the length of a side.

While it's true that the problem doesn't give the length of a side, I might ask Lauren if there is any way we could figure that out. It might just be that she

Joe
age 15

Interpretation
Novice

The answer is 52.04

I used the 1,1,sqare root of two formula

Joe is assuming that the 4.6 is the hypotenuse of a 45-45-90 triangle, and divides by 2 to find the unknown length of the bottom leg. He doesn't notice, however, that his length for that side doesn't match the given length of the other side! I might simply point that out to him – if it is indeed a 45-45-90 triangle, don't those two things need to be equal?

Sandy
age 15

Interpretation
Novice

What is wrong is the hypotenuse of the triangle, because it should really be about 5.66 cm. This would make the perimeter= to aproximately 218.51 cm. EXTRA: another thing that could be wrong is the fact that there isnt enough information on the triangle, so maybe the triangle is equilateral. This would make the perimeter = to aproximately 135.76 cm.

To find out the hypotenuse I did the following: First I noticed that the triangle that was formed was a right triangle, so it is obvious that it is going to be a 45-45-90 triangle. Then I used the formula to find the lenghts of the sides of a 45-45-90 triangle wich is:

45º 45º 90º

1 1 one, square root of 2.

so then the sides would be, 4, 4 and 4 square root of 2, so: sides= 4 cm, 4 cm, and 5.656854 cm, which I estimated to be 5.66

Then to find the perimeter I just did.. 5.65 + 4 + 4 x 2, and that gave me the perimeter of the whole triangle.and then I multiplied that x 8 and it gave me 218.509664 which i estimated it to be 218.51 cm.

FOr the extra I thought, if the 3 sides are the same the perimeter would of the whole triangle would be = to 5.656854 x 3 = 16.970562 and then you multiply that times 8 and the perimeter of the octagon is equal to 135.764496 , which i estimate to be around 135.76 cm.

Sandy has made an assumption about the triangle shown (or else really thinks that all right triangles are 45-45-90 triangles). She has also done the wrong thing to find the perimeter of the octagon.

I might start by asking her how she knows that's a 45-45-90 triangle. I would address the perimeter issue later.

Tom
age 15

Interpretation
Apprentice

The perimeter would be 37.86cm.

a2+b2=c2

a2+42=4.62

a2=5.6

a2=2.36

2.36*16=37.86

I don't know what's wrong with the picture.

Tom has simply found the perimeter as if there's nothing wrong. That's exactly what the Texas state test expected students to do. But while he indicates that he knows he's supposed to find something wrong, he hasn't said anything more about it. I would ask him if he knows how to find the measures of the angles of the right triangle with which he's working.

Azi
age 13

Interpretation
Apprentice

This is impossible regular hexagon because the triangle formed does not satisfy the requirements of a 3-60-90 right triangle.

2. If it were to be corrected the perrimeter would be 27.6

1. This is an impossible regular hexagon due to the fact that when a hexagon is broken into 8 right triangles they are each 30, 60, 90 degree triangles and thus the formula for their sides is x, x square root of 3, and 2x. The triangle formed in this example does not follow these requirements because 4 does not equal 2.3 * the square root of 3.

2. In order to make this hexagon feasible one would have to make the apothem of the hexagon equal to approximately 3.9, and the shortest side of the triangle 2.3. Thus the perimeter of the hexagon would equal 27.6.

Azi has some really good ideas, and some sound mathematical statements. Unfortunately, he's mistaken the figure for a hexagon instead of an octagon. I would point that out to him and ask him to see how that changes his work.

Guna
age 12

Interpretation
Apprentice

My solution is that the apoteum of the regular octagon is not 4. The perimeter is 36.8.

First I noticed that it was a regular octagon, meaning that the sides are all equal. Then I divided the base of the octagon, which is 4.6, making half of it (2.3). After I did the Pythgorean Theorem, I found out that the height is really not 4 but the height is about 3.9. so I knew what was wrong with the picture. Then you would multiply 4.6 by 8 to get the perimeter.

It looks like Guna has misread the problem, or perhaps mislabled her picture. Either that or she thinks that the side of the octagon equals the radius. I would ask her how she knew that side was 4.6, since it's not labeled as such (but that other side is).

Effat
age 15

Interpretation
Apprentice

the perimeter of the octagon is 28.16 cm.

Since we know that all angles of an octagon add up to 1080, one angle of octagon is 135.

A line from center to one vertex bisects the angle of octagon so now one angle is 67.5 cm.Another line from center to other vertex bisects the other angle so the other angle is also 67.5 cm. Now we have a triangle with two angles 67.5, & 67.5. Two find the third angle of the triangle, we need to subtract both angles from 180.

third angle = 180 - 67.5 - 67.5

third angle = 45

Effat has done a nice job of finding a possible perimeter. But he hasn't said what must be wrong in the given picture. I would just ask him if he could explain that.

We draw altitude which bisects the triangle & divides it into two equal parts.Now we have a right triangle with angles of 90, 67.5, and 22.5 . In order to find the perimeter of the octagon, we need to find one side of the octagon which is twice the length of base of the right triangle.To find the base length,

$$\cos 67.5 = \frac{X}{4.6}$$

$$\cos 67.5 * 4.6 = X$$
$$1.76 = X$$

The base of the right triangle is 1.76 cm.One side of octagon is twice the length of base of right triangle so

one side of octagon = 2 * 1.76

one side of octagon = 3.52

THEN

Perimeter of the octagon = 3.52 * 8

Perimeter of octagon = 28.16 cm.

Anthony
age 15

Interpretation
Practitioner

1) the problem is that the hypotenuse is suposed to be equal to 4.3cm.

2) The perimeter of the octogon is 25.6cm.

In order to find the actual hypotenuse i used basic trigonometry by finding the sine of the angle opposite the apothem which is 1/2 of the interior angle of the octogon. in doing so i found that the hypotenuse was actually equal to 4.3 cm.

After that i used pythagorean's theorem to find the length 1/2 the side of the octogon. by that i mean that since the apothem, the radius, and 1/2 of the base of the octogon forms a right triangle, then the pythagorean theorem should be able to help me find 1/2 of the base. i multiplied 1/2the base(1.6)by 2 in order to find the entire base and multiplied the entire base(3.2) by 8 (the amount of sides in an octogon) and found the perimeter to be 25.6cm.

Anthony has done a nice job with the math. I would ask him to add some more detail so that another student could follow his work. For example, what angle measures did he use with sine?

Zach
age 14

Interpretation
Expert

1. The length of the apothem of the regular octagon is incorrect.

2. Fixing the length of the apothem will result in a perimeter measuring 28.16 cm

First, I noted that the octagon is regular, meaning that all sides are congruent; this is crucial in finding the perimeter. Then, I used the equation 180(n-2)=m; n being the number of sides in a regular polygon and m being the sum of interior angles. Plugging 8 in for n results in the sum of interior angles of being 1080 degrees. The next step is to divide 1080 by 8, which equals 135. 135 is the measure of each interior angle.

Now, if you can imagine a circle that goes around the octagon with each vertice on the edge of the circle. If the circle shares the same center as the octagon, then each diagonal that goes to the angles of the octagon is a radius of the circle. That means that each diagonal is congruent. The angle created by the diagonal will have to equal the same apothem on either side. This is because the diagonals are congruent, along with the right angles and the apothem! In other words, the angle will have to be

Zach has solved the problem and also the Extra. And he’s written a very nice explanation! Note, however, that he questions the clarity of his writing. I’d say he’s definitely headed in the right direction!

bisected. So, in the triangle with the apothem, we have a hypotenuse of 4.6, an apothem of x, and angles measuring 67.5 and 22.5. We can use sin 67.5=x/4.6; x being the apothem and opposite side of 67.5. Solve the equation:

Sin 67.5=x/4.6

4.6 sin 67.5= x

We get the answer of about 4.25 cm for our apothem, which is not the length of the apothem in the question. This is what is wrong.

Now, use the Pythagorean Theorem to find the last side of this triangle. 4.6^2-4.25^2=the last side. This equals 1.76 cm for the last side of the triangle.

The triangle next to the triangle we have is congruent by SAS; or apothem, top angle which is 22.5 (this is the same because you use the same method as the other triangle to find the angles), and the diagonal or radius of the circle.

If the triangles are congruent, then by CPCFC(congruent parts of congruent figures are congruent) the last side is congruent. Now, one side of the octagon is 1.76x2 or 3.52. Because the octagon is regular, we can multiply this number by 8, which results in the measure of the perimeter: 28.16 cm.

EXTRA:

If we say that the apothem is correct, then the diagonals are incorrect. We can find their measure using the techniques we used before (Bisected interior angle, law of sines, Pythagorean theorem) The diagonal then is approximately 4.3 cm. The last side of the triangle is then 1.58 which we can multiply then 2 and then by 8. This results in the perimeter being approximately 25.28 cm.

This problem really showed me the importance of being able to explain things. I knew how to solve it, but putting it in words is extremely hard! I hope with practice I will be able to do so more clearly.

