# Algebra PoW Packet

# *That's Interesting!*

**April 14, 2008 • http://mathforum.org/algpow/**

## Welcome!

We hope that you are finding these packets useful and informative. We would really appreciate hearing from you about them. In particular, what has been most useful to you? What would you like to see added or changed?

You can access the discussion group via the link to "PoW Members" in your Teacher Office, or go to algpow-teachers directly at http://mathforum.org/kb/forum.jspa?forumID=528 . Login to the discussions using your PoW username and password.

This packet contains a new problem, the "answer check," our solution and scoring rubric, a note about possible common mistakes we may see, and ideas for implementing the problem in the classroom.

## The Problem

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*That's Interesting!* grew out of thinking about exponential functions and exponential growth or decay. While most adults are familiar with the idea that investing money leads to great gains in the long run, most kids are not yet worried about such things. This problem might prove somewhat eye-opening to them, and you might want to encourage some reflection on the results once they've solved it. A natural extension question that may arise is how much money Bill would have if he had continued to invest $1000 per year into his account instead of stopping after the initial investment. While that's a much more complicated calculation, if you have students who are comfortable using a spreadsheet it can easily be determined and will probably add to their surprise about the power of investing. Further extension with a spreadsheet might include trying different interest rates to see what effect that has on the long-term gain.

The problem gives kids practice in producing an expression or equation that models a given situation, in this case the amount of money each man will have after x years. Because of the exponential nature of the problem, solving question 4 and the Extra algebraically will most likely be beyond them since setting the two amounts equal and solving for the break-even point would require the use of logarithms. But students can graph the two functions or make a table to compare them, or might just apply their general knowledge of exponential and linear growth to draw a conclusion.

While we're not trying to turn your kids into young capitalists, *That's Interesting!* provides a pretty vivid demonstration of the power of exponential growth. Let us know how they make out with it.

## Answer Check

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After students submit their solution, they can choose to "check" their answer by looking at the answer that we provide. Along with the answer itself (which never explains how to actually *get* the answer, and in this case only gives two of the four answers), we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

## Our Solution

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In the past we have only made our solutions available to mentors, as we felt using authentic student work provided better examples of how they actually solve the problem. However, we also realized that sometimes having the solution ahead of time could be helpful, as we often include tips for how to support students in different areas of their work, or in thinking about different parts of the problem. When appropriate, we also include multiple ways to solve the problem.

## Scoring Rubric

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The problem-specific rubric is something we write for every problem for use by those who are assessing student work. It spells out what we expect from students in three areas of problem solving and three areas of communication. It is designed to give students targeted information on their strengths and weaknesses. It's not intended to produce a "grade", but rather help students begin to think about where they're strong and where they could improve. You might share the problem-specific rubric with your students, or you might prefer to give them the generic rubric that's available from the Teacher Documents page at http://mathforum.org/pow/teacher/.

**Common Mistakes** right here!

We haven't used this problem with students, so we can't say for sure what students might find confusing. But we can make some educated guesses.

Some students may struggle with the interest calculation, or the timing of it. We've constructed the problem slightly unrealistically with the full 10% being paid on the last day of the year in an effort to keep it simpler. The timing of the deposits and interest as the day before the amounts are evaluated will hopefully remove potential confusion about balances midway through a calendar year.

Students might also struggle with question 4 as they will most likely not have the knowledge to set the two men's balances equal and solve for a break-even point. We tend to avoid writing problems that rely on graphing to solve because it's harder for kids to submit their graphs with their solution online, so we've left this a little open in terms of not asking exactly when Bill might overtake Curtis but simply whether or not he does. Note that images can be submitted online, so students who work in a spreadsheet or use a graphing utility can take a screen-shot of their work and then convert it into a standard image format and upload it with their work. Details on uploading images can be found near the bottom of the submission page.

**Good luck!**

We're excited about providing these new resources to you and hope you find them useful. We'd love to get feedback and ideas from you on the algpow-teachers discussion group.

– Riz

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**Problem**

## That's Interesting!

On December 31, 2007, Curtis and Bill each had $1000 to start saving for retirement. The two men had different ideas about the best way to save, though.

Curtis, who doesn't trust banks, put his money in a can and buried it in his back yard. He plans to continue adding $1000 to the can on the last day of each year until he is ready to retire.

Bill invested his $1000 in a bank account that will pay 10% interest annually on the last day of the year. Unlike Curtis, he does not plan to continue investing more money each year. He figures that with the interest he will earn his bank account will eventually be worth more than the money in Curtis's can.

1. How much money will each man have on January 1, 2009?
2. How much money will each man have on January 1, 2010?
3. Write a variable expression for each man that represents how much money he will have x years after January 1, 2008.
4. Will Bill's bank account ever be worth more than the money Curtis has collected in his can? How do you know?

**Extra:** Suppose that instead Bill had invested $2000 and Curtis had buried $500 on December 31, 2007. If Bill invests no more money and Curtis continues to bury $500 each year, for what calendar years would Curtis have more money on January 1st? How do you know?

## Answer Check

On January 1, 2009, Bill has $1100 and Curtis has $2000. On January 1, 2010, Bill has $1210 and Curtis has $3000. Be sure that you've also answered questions 3 and 4 and explained clearly how you came up with your answers.

If your answers **don't** match ours,

* did you understand that on January 1, 2009 Curtis has just buried another $1000 the day before on December 31, 2008?
* did you remember that each year Bill earns 10% of his current account value, not just another 10% of the original $1000?
* did you understand that if x is the number of years after January 1, 2008, then x = 1 in 2009, x = 2 in 2010, and so on?
* did you try making a chart to look for patterns in how much money each man has each year?
* did you try graphing your expressions or equations?
* did you check your math?

If any of these comments help, revise your solution. If you're still stuck, do the best you can and write a note telling us what you've tried and where you are stuck.

If your answers **do** match ours,

* have you also answered questions 3 and 4?
* have you clearly shown and explained the thinking and work you did?
* did you make any mistakes along the way? If so, how did you find and fix them?
* are there any hints that you would give another student?
* are you confident that you could solve another problem like this successfully?
* have you tried the Extra question?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did - you might answer one or more of the questions above.

## Our Solution

*Question 1*

The problem starts on December 31, 2007 with each man investing $1000. On December 31, 2008, Curtis will bury another $1000, meaning that a day later on January 1, 2009, he will have $2000 in his can.

Bill will earn 10% interest on December 31, 2008. I took $1000 times 10% to see how much interest that is. 1000(0.10) = 100, so he earned $100 in interest. That means one day later on January 1, 2009, he will have $1000 + $100 or $1100 in his account.

Another way to calculate the interest is to multiply his balance by (100% + 10%) since the 100% keeps the amount he has and the 10% adds the interest. Turning (100% + 10%) into decimals I got 1 + 0.10 or 1.10, so his balance will be $1000(1.10) or $1100.

*On January 1, 2009, Curtis has $2000 and Bill has $1100.*

*Question 2*

On December 31, 2009, Curtis puts another $1000 in his can, so a day later on January 1, 2010, he will have $3000.

Bill earns 10% of his $1100 balance on December 31, 2009. As in question 1, I can calculate that two ways. First, find 10% of 1100 and then add the interest to the $1100 balance:

1100(0.10) = 110, and 1100 + 110 = 1210.

Second, find 110% of 1100:

1100(1.10) = 1210.

*On January 1, 2010, Curtis has $3000 and Bill has $1210.*

*Question 3*

Let x = the number of years after January 1, 2008. That means in 2008 x will be 0, in 2009 x will be 1, in 2010 x will be 2, and so on.

I made a table to help me organize my thoughts and see what was going on:

| Year | x | Curtis | Bill |
|---|---|---|---|
| 2008 | 0 | 1000 | 1000 |
| 2009 | 1 | 2000 | 1000(1.10) = 1100 |
| 2010 | 2 | 3000 | 1000(1.10)(1.10) = 1210 |
| 2011 | 3 | 4000 | 1000(1.10)(1.10)(1.10) = 1610.51 |

Once I looked at the data in this form, I saw some patterns. Every time a year goes by x increases by 1 and Curtis adds another $1000. The amount in his can each year is just $1000 times 1 more than x, so an expression for his amount is 1000(x + 1) or 1000x + 1000.

For Bill, every time a year goes by his balance gets multiplied by (1.10) to do the 10% increase. I noticed that the number of times the original $1000 gets multiplied by 1.10 is the same as the value of x. An expression for his amount is 1000(1.10)^x or just 1000(1.1)^x.

*So, x years after January 1, 2008, Curtis will have 1000x + 1000 dollars and Bill will have 1000(1.1)^x dollars.*

*Question 4*

There are several ways a student might answer question 4. I'll show the ones I've thought of here but kids might come up with others. As long as their thinking is sound and well explained, it should be accepted.

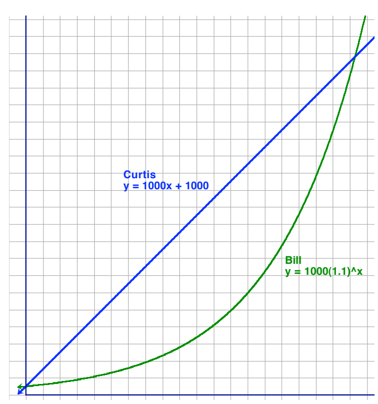
*Method 1 - use knowledge of linear and exponential functions*

I recognize that Curtis's expression is linear in the form y = mx + b. He starts with $1000 (the y-intercept) and gains another $1000 each year (the slope). I also recognize that Bill's expression is exponential, in the form y = ab^x. With the value of b as 1.1 the function will increase over time. With the value of a as 1000 he starts with $1000 (the same y-intercept as Curtis).

I know that exponential functions start growing slowly, and so after the first year Bill falls behind Curtis, and continues to fall further behind as the early years go by, as shown in my chart from question 3. But I know that at some point exponential growth becomes so rapid that the graph moves toward being almost vertical, and I'm confident that eventually it will intersect with Curtis's line and move above it. I'm not sure how long that will take and if both men will still be alive, but I'm sure it will eventually happen.

*Method 2 - use a graphing utility and compare the graphs*

I wrote both men's expressions as equations and then graphed them. For Curtis I used y = 1000x + 1000 and for Bill I used y = 1000(1.1)^x. My graph showed that while Bill falls behind at first, he eventually catches and passes Curtis. I've included the graph:



*Method 3 - make a table or use a spreadsheet*

I used a spreadsheet to extend the table I started in question 3. I rounded all the dollars to whole numbers. I found that Bill eventually catches and passes Curtis in terms of the total amount of money. Here's my table:

| Year | x | Curtis | Bill | Year | x | Curtis | Bill |
|---|---|---|---|---|---|---|---|
| 2008 | 0 | 1000 | 1000 | 2029 | 21 | 22000 | 7400 |
| 2009 | 1 | 2000 | 1100 | 2030 | 22 | 23000 | 8140 |
| 2010 | 2 | 3000 | 1210 | 2031 | 23 | 24000 | 8954 |
| 2011 | 3 | 4000 | 1331 | 2032 | 24 | 25000 | 9850 |
| 2012 | 4 | 5000 | 1464 | 2033 | 25 | 26000 | 10835 |
| 2013 | 5 | 6000 | 1611 | 2034 | 26 | 27000 | 11918 |
| 2014 | 6 | 7000 | 1772 | 2035 | 27 | 28000 | 13110 |
| 2015 | 7 | 8000 | 1949 | 2036 | 28 | 29000 | 14421 |
| 2016 | 8 | 9000 | 2144 | 2037 | 29 | 30000 | 15863 |
| 2017 | 9 | 10000 | 2358 | 2038 | 30 | 31000 | 17449 |
| 2018 | 10 | 11000 | 2594 | 2039 | 31 | 32000 | 19194 |
| 2019 | 11 | 12000 | 2853 | 2040 | 32 | 33000 | 21114 |
| 2020 | 12 | 13000 | 3138 | 2041 | 33 | 34000 | 23225 |
| 2021 | 13 | 14000 | 3452 | 2042 | 34 | 35000 | 25548 |
| 2022 | 14 | 15000 | 3797 | 2043 | 35 | 36000 | 28102 |
| 2023 | 15 | 16000 | 4177 | 2044 | 36 | 37000 | 30913 |
| 2024 | 16 | 17000 | 4595 | 2045 | 37 | 38000 | 34004 |
| 2025 | 17 | 18000 | 5054 | 2046 | 38 | 39000 | 37404 |
| 2026 | 18 | 19000 | 5560 | 2047 | 39 | 40000 | 41145 |
| 2027 | 19 | 20000 | 6116 | 2048 | 40 | 41000 | 45259 |
| 2028 | 20 | 21000 | 6727 | 2049 | 41 | 42000 | 49785 |

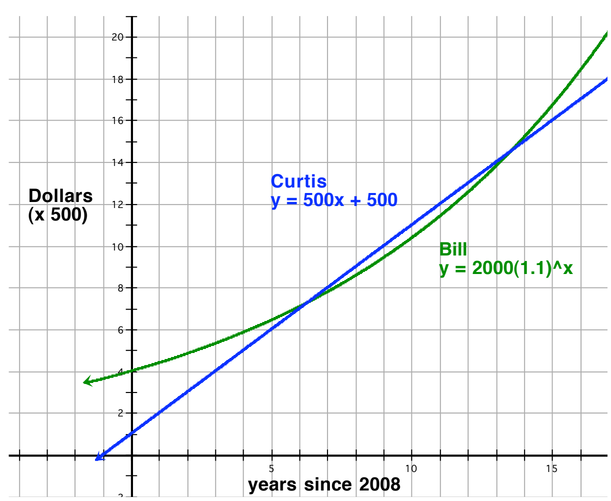
*Yes, Bill's bank account will eventually be worth more than all the money Curtis has put in his can.*

**Extra**

Again there are at least two possible approaches to solving the Extra, including graphing or making a table as in question 4 above.

*Method 1 - use a graphing utility to compare the graphs*

Since Curtis is now putting $500 in the can each year, the equation for his total money is y = 500x + 500. Bill now starts with $2000, so his equation is y = 2000(1.1)^x. I graphed the two equations so I could compare how much they had each year:




Since Bill deposits more than Curtis at the start, he has more in 2008 (at x = 0 on the graph). But since the exponential growth starts slowly, Curtis catches and passes him by x = 7, which is 2015. Then Bill comes back up and passes Curtis by x = 14, which is 2022. From then on, as I saw in the problem, Bill will continue to increase his lead.

Curtis has more money than Bill on January 1 from 2015 to 2021.

*Method 2 - make a table or use a spreadsheet*

I made a spreadsheet like the one in the problem to track how much money each man had each year. For Curtis my equation was 500 + 500x since he puts in $500 each year. For Bill it was 2000(1.1)^x since he deposits 2000 at the start and earns 10% interest. Here is my chart:

| Year | x | Curtis | Bill |
|---|---|---|---|
| 2008 | 0 | 500 | 2000 |
| 2009 | 1 | 1000 | 2200 |
| 2010 | 2 | 1500 | 2420 |
| 2011 | 3 | 2000 | 2662 |
| 2012 | 4 | 2500 | 2928 |
| 2013 | 5 | 3000 | 3221 |
| 2014 | 6 | 3500 | 3543 |
| 2015 | 7 | 4000 | 3897 |
| 2016 | 8 | 4500 | 4287 |
| 2017 | 9 | 5000 | 4716 |
| 2018 | 10 | 5500 | 5187 |
| 2019 | 11 | 6000 | 5706 |
| 2020 | 12 | 6500 | 6277 |
| 2021 | 13 | 7000 | 6905 |
| 2022 | 14 | 7500 | 7595 |
| 2023 | 15 | 8000 | 8354 |
| 2024 | 16 | 8500 | 9190 |
| 2025 | 17 | 9000 | 10109 |
| 2026 | 18 | 9500 | 11120 |
| 2027 | 19 | 10000 | 12232 |
| 2028 | 20 | 10500 | 13455 |

Bill starts out with more since he deposits more than Curtis at the beginning, but by 2015 Curtis has passed him. He holds the lead until 2022 when Bill passes him back, and then Bill continues to pull away, as he did in the main problem.

Curtis has more money than Bill on January 1 from 2015 to 2021.

# Algebra Problem of the Week Scoring Rubric for *That's Interesting!*

For each category, choose the level that *best describes* the student's work.

| | Novice | Apprentice | Practitioner | Expert |
|---|---|---|---|---|
| **Problem Solving** | | | | |
| **Interpretation** | shows little understanding of the concepts involved - see the Practitioner column | shows understanding of most but not all of the concepts in the Practitioner column<br><br>(for example, understands simple interest formula but is confused by the timing of the deposit and interest payments)<br><br>does not complete all parts of the problem | understands what 10% annual interest means and how to calculate it, including that the amount of interest will increase each year as the balance grows<br><br>understands that the money is buried and interest is paid on the last day of the year and the amounts in questions 1 and 2 are calculated on the first day of the year, before any interest is earned for that year<br><br>attempts to answer all the questions | solves the main problem and the Extra correctly, and is at least a Practitioner in Strategy |
| **Strategy** | has few ideas that will lead them toward a successful solution | picks an incorrect strategy, or relies on luck to get the right answer | picks a sound strategy—success achieved through skill, not luck<br><br>after producing the two expressions in question 3, might solve question 4 by graphing them to compare the amounts over time, making a table, or applying general knowledge of exponential and linear functions | uses two separate strategies or an unusual or sophisticated strategy<br><br>(might set the expressions equal and solve the equation by using a logarithm to determine the break-even point) |
| **Accuracy** | work contains many errors | work is mostly accurate, with a few errors | work is accurate and contains no arithmetic mistakes | generally not possible - can't be more accurate than Practitioner |
| **Communication** | | | | |
| **Completeness** | has written very little that tells or shows how they found their answer | explains the steps used to find the answer but shows very few of the calculations and work<br>OR<br>shows the work but does not explain the thinking behind it<br><br>does not define variable(s) | defines variable(s)<br><br>explains all of the steps taken to solve the problem<br><br>shows and explains equations, formulas, and calculations used | adds in useful extensions and further explanation of some of the ideas involved<br><br>the additions are helpful, not just "I'll say more to get more credit" |
| **Clarity** | explanation is very difficult to read and follow | explanation isn't entirely unclear, but would be hard for another student to follow<br><br>explanation is long and is written entirely in one paragraph<br><br>explanation contains many spelling and typing errors | explains all of the steps in such a way that another student would understand<br><br>makes an effort to check their formatting, spelling, and typing (a few errors are fine as long as they don't make it hard to read) | formats things exceptionally clearly<br><br>answer is very readable and appealing |
| **Reflection** | The items in the columns to the right are considered reflective. They could be in the solution or the comment left after viewing the Math Forum's answer. | checked answer in some way (in addition to viewing the answer provided by the Math Forum)<br><br>reflected on the reasonableness of their answer | connected the problem to prior problems or experiences<br><br>explained where they are stuck<br><br>summarized the process they used | commented on and explained the ease or difficulty of the problem<br><br>revised and improved their work |
| | did nothing reflective | did one reflective thing | did two reflective things | did three or more reflective things *or* did an exceptional job with two of them |