Problem Solving and Communication Activity Series

Lesson 1: Understanding the Problem

What does it mean to fully understand a problem, and how does it help students find solution paths and build confidence? Understanding the problem is the first principle of problem solving described in mathematician George Polya's (1945) *How to Solve It*. It is also related to math educators Stephen Krulik and Jesse Rudnick's (1987) first problem-solving stages, read and explore, described in their book, *Problem Solving: A Handbook for Teachers*. Included in this document are several activities that support students to develop strategies for understanding challenging math problems, along with facilitation suggestions for teachers.

Goals for the Understanding the Problem strategy

Different techniques for understanding the problem can lead us to ideas for solving a problem we have never used before. Good problem-solvers use this problem-solving strategy and may come back to it often as they're working on the problem, to refine their strategy, see if they can find better solutions, or find other, even more interesting questions. Specifically, good problem-solvers:

- Use various methods to make sense of a problem from different perspectives.
- Pull out the relevant information.
- Figure out what counts as an answer.
- Connect to prior knowledge and experience.
- Identify questions and relationships that make the problem interesting.

Writing Goals

Writing is an integral part of understanding the problem and builds momentum in thought. It helps the problem solver organize their information and articulate the questions they will address on their journey towards the solution. Specifically, when trying to understand the problem, good problem-solvers might write:

- Create an organized list of what they noticed and what they wondered about the problem.
- Paraphrase the problem in their own words.
- Question or hypothesize, focusing their thinking on specific parts of the problem (e.g., I'll know I'm right when…; I could solve the problem if …; the part that makes it hard is …; I'll need to use the fact that …)

Activities

I. Noticings/Wonderings

Format: pairs/small groups, big-group brainstorming or go-round activity, or think-pair-share.

Make a list of all of the mathematical information and relationships you notice about the problem, and everything you wonder about the problem. Your noticings may include:

- The quantities that can be counted or measured.
- Relationships between quantities.
- Information that is not given in the problem.
- Key words from the problem.

Your wonderings may include:

- I wonder what will happen if …
- I wonder what this word means …
- I wonder if this pattern will continue …

Sample Activity: Forget the Question

Present the problem to students without the question. As a whole class, or in small groups, have students go around quickly and offer one mathematical relationship or bit of mathematical information that they notice, or a mathematical question they are wondering about. Record each noticing and wondering for later use. Continue around the group until no one has anything more to offer.

> ***Note:*** It is useful to refer back to and update your class list of noticings/wonderings throughout the process of understanding the problem and solving the problem. Students should be encouraged to keep their list, and/or the class list, available and update it as they work.

Key Outcomes

- Student ownership and understanding of the question to be solved.
- Momentum toward a solution path stimulated by all of the mathematical quantities and relationships noticed.
- Slowing down the thinking process and surfacing all of the information and questions that are too easily passed over or dismissed.
- Articulation of specific sub-problems or questions students need to answer or learn more about in order to solve the problem.
- Identifying other questions and features of the problem that may be even more interesting and challenging for students.

II. Extracting the Information and Question

Format: whole group brainstorming, individual brainstorming, or think-pair-share.

List as mathematically and succinctly as possible the key information that may be useful in solving the problem and state what will count as an answer.

- Identify and list important information given in the problem:
  - What quantities are given?
  - What terms are important?
  - What constraints are given?
- Predict as much as you can about the final answer:
  - What will the units of the solution be (what will be counted or measured)?
  - What justification is needed/what am I trying to prove?
  - Can I figure out upper and lower bounds?
  - Could the answer be negative? Could it be a non-integer?

Sample Activity: PoW IQ

"PoW IQ" stands for extracting the **I**nformation in the problem, and understanding the **Q**uestion.

Have each student make a table with very compact mathematical information from the problem in the left column, calculations or mathematical relationships they see in the middle column, and questions they should explore for the final answer in the right column.

Key Outcomes

- Student ownership and understanding of the constraints a full solution requires.
- Articulating mathematical information in a simple, compact format that makes patterns and relationships visible and moves students toward possible solution paths.

III. Paraphrasing

Format: think-pair-share or students working individually.

The goal of paraphrasing a problem is to have students analyze the language of a problem and make clear the mathematics behind the problem situation. Some prompts you might use with students are:

- Identify the key words/phrases in the problem. How would you define them?
- How would you rewrite the problem?
- Same math idea/Different math story: How could you put the problem in a different scenario, while preserving the math behind the problem?

Sample Activity: In Your Own Words

Read the problem as a group. Discuss what the words mean, make sense of the situation and of the question that must be answered. Put away the problem and any notes and individually paraphrase the problem, using one of the prompts above.

Key Outcomes

- Learn to make sure the problem makes sense to you and that you know what has to be figured out.
- Put your thoughts in writing so you can compare your thinking to the original problem statement and see what you may be missing or changing without realizing it.

IV. Acting it Out

Format: whole group with a few volunteers to act it out and the rest advising, or in small groups.

This approach often requires the most teacher/expert support to ensure the tools or manipulatives that support the investigation are available, and that students are making sure their modeling of the problem fits the necessary constraints. Some problems do not easily lend themselves to this approach. Acting the problem out would probably be used in tandem with one of the activities above, since it doesn't end in a written representation of students' understanding of the problem. Some examples include:

- Physically acting out the problem by using actual materials from the problem situation or using virtual manipulatives (You might act out a simpler version of the problem, for example, using smaller numbers).
- Drawing a rough sketch (This is different from drawing a picture as a strategy to solve the problem, since you know you might be drawing it imperfectly and less representative, but you are just trying to get a sense of relationships).
- Doing the problem "wrong": similar to a quick version of guess and check, doing the problem wrong can refer to working through the problem by guessing a number for an unknown quantity, or trying to find an answer that works without being sure you have found every possible answer. In either case, the focus is on understanding the problem scenario and key relationships, rather than trying to get a full solution to the problem.

Sample Activity: Try It!

As a group, students and the teacher work together to decide on ways to represent the key objects or concepts in a problem. For example, the teacher might provide one or several virtual manipulatives or software tools, or the teacher might provide several types of physical manipulatives. The students then represent different object in the problem, and try to model the situation. Students in the "checker" role see if the model responds to the constraints of the problem that had been noticed and if not, suggest what might be wrong. As the students act out the problem, designated scribes add to the students' list of noticings/wonderings.

Key Outcomes:

- Use visual and physical intelligence to develop a sense of what is going on in the problem.
- Figure out an answer or a good estimate and use this to start thinking of explanations about why it works that way.

Examples: Cadence (PreAlgPoW)

The goal of these lessons is for the students to experience generating their own understanding of the problem. While it's tempting to steer them towards certain key ideas, we want students to experience the gain in confidence that comes from being able to rely on their own resources in order to get going. As a result, we tend to hold back on suggestions and focus on supporting the student's own thinking. If students are stuck, or we feel their ideas need further probing and clarifying, we might help with facilitating questions that reinforce the problem-solving strategies. Check out the "prealgpow-teachers" discussion group (http://mathforum.org/kb/forum.jspa?forumID=527) for conversations about this problem in which teachers can share questions, student solutions, and implementation ideas.

If we do facilitate by asking some strategy questions, then at the end of the session we often ask students to notice the questions and suggestions we asked so that they can begin to do that for themselves: Which were helpful? Could you see how you could use these with other problems? Which questions would you put on a class list of "Ways to get Unstuck in Understanding the Problem"?

Below you'll find examples of student thinking, both to provide illustrations of the activities we described above, and to anticipate student thinking that may come up in your class that you might want to probe or question further.

I. Noticings/Wonderings

Noticings/Wonderings students may generate:

- Short friend says she is faster. Tall friend says she and her friend have the same speed.
- Tall friend has a step size of 38 inches.
- Short friend has a step size that is 2/3 of 38 inches.
- Both travel 100 feet in 20 seconds.
- One problem asks for feet per minute, and one asks for steps per minute.
- Both feet and inches are used in the problem.
- I wonder if feet per minute can be different from steps per minute? Under what conditions?
- I wonder if smaller steps will lead to a faster or slower cadence?
- I wonder which meaning of "faster" each friend is thinking about?
- I wonder whether I should round steps and step lengths up or down or whether I should be very precise?

Noticings/Wonderings that might indicate possible misunderstandings:

- Whoever takes the fewest steps has the fastest cadence.
- I notice that the shorter person has a lower center of gravity and less mass.
- I notice that they go the same speed, so they take the same number of steps per minute.
- I notice that cadence refers to who takes the most steps (not the most steps *per minute*).
- I notice the problem already says they go the same speed, so I wonder why the question asks about cadence (steps per minute)?

II. Extracting the Information and Question

PoW IQ notes students may generate:

Information	Calculations	Questions
My step length = 38 inches Friend's step length = (2/3) * 38 in. Both of us go 100 ft/20 sec	100 feet = 1200 inches 20 sec is (1/3) of a minute Friend's step length is 25 and (1/3) in. (# of steps/100 ft) * (# of feet/min) = cadence in steps/min	Find my steps per 100 feet. Find my friend's steps per 10 feet. Find both speeds in feet/min (convert from 100 ft/20 sec). Find both cadences (i.e., steps/min).

PoW IQ notes that might indicate possible misunderstandings:

Information	Calculations	Questions
Because our rates are the same, we take the same number of steps. Distance per step, times distance per minute, gives cadence.	100 ft/20 sec = 5 ft/sec 38" = 3.167' (2/3) * 3.167' = 2.11' (5 ft/sec) * 3.167 feet = 15.835 steps/min (5 ft/sec) * 2.11 feet = 10.556 steps/min	Why are our rates the same if we have different stride lengths? How much further do I travel than my friend, if we do the same speed for the same time?

III. Paraphrasing

Key Words

Key words students may generate:

- Faster cadence means more steps per minute.
- Faster speed means more feet per minute.

Key words that might indicate possible misunderstandings:

- Faster cadence means which friend takes the most steps (not thinking of cadence as rate)
- Faster cadence means which friend takes the fewest steps per minute.
- Faster cadence means which friend travels the most feet per minute.
- Faster speed means which friend takes the most steps per minute.

In your own words

Paraphrases students may generate:

Two friends walk at the same rate in feet/minute, but one of them takes shorter steps. Which friend has the faster cadence? Can I say which friend is "faster?"

Paraphrases that might indicate possible misunderstandings:

- Two friends travel the same number of feet/minute. Which one takes the most steps?
- Figure out how many steps each person takes to go 100 feet. Then figure out how many steps they take to go 300 feet. That tells you which friend has the faster cadence.

Same math idea/Different math story

Stories students may generate:

Two people riding mountain bikes are in different gears. One of them goes 38" per pedal revolution, and the other one goes 2/3 of that in one pedal revolution. If both bikers go 100 feet in 20 seconds, which one is pedaling faster?

Stories that might indicate possible misunderstandings:

- Michael Phelps swims 38 inches per second, and Ian Crocker swims 2/3 as much as that per second. Who has the faster cadence?
- Michael Phelps can swim 50 meters in only 38 strokes, and Ian Crocker takes 2/3 as many strokes. Who has the faster cadence?

IV. Acting it out

Ways to act out the problem students may generate:

- Find a tall student and a short student and have them walk a measured distance, trying to finish at the exact same time. Try to observe who has the faster cadence.
- On your own, try to walk a measured distance at a constant speed. On the first try, take long steps. Then try again, this time taking short steps. What do you notice?
- Using technology variation: use a CBR/CBL to help you walk at a constant speed. Use the motion sensor to plot your distance versus time graph. Try to match it to a graph of someone moving at a constant speed.

Ways to act out the problem that might indicate possible misunderstandings:

- Have two people, one taller and one shorter, walk the same distance and observe who finishes first.
- Have two people take different numbers of steps in a minute and observe who goes farther.
- See who is faster, a tall student taking quicker steps, or a tall student taking slower steps?

References

Krulik, S., & Rudnick, J. A. (1987). *Problem solving: A handbook for teachers* (2nd ed.). Boston: Allyn and Bacon.

Polya, G. (1945). *How to solve it*. Princeton, New Jersey: Princeton University Press.