



# Algebra PoW Packet

## *Alvin's Theorem*

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### Welcome!

We hope that you are finding these packets useful and informative. We would really appreciate hearing from you about them. In particular, what has been most helpful to you? What would you like to see added or changed?

You can access the discussion group via the link to "PoW Members" in your Teacher Office, or go to [algpow-teachers](http://mathforum.org/kb/forum.jspa?forumID=528) directly: <http://mathforum.org/kb/forum.jspa?forumID=528>. Login to the discussion using your PoW username and password.

This packet contains a problem from the Library, the "answer check," our solution and scoring rubric, a note about possible common mistakes we may see, and some student solutions.

### The Problem

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We are revisiting *Alvin's Theorem*, Problem 3052 from the Library. It's about using a general case argument to support an observation a student makes about the difference of consecutive square numbers. Many students do not understand the idea of a general case argument, and assume that providing several specific examples is enough to show that something will always be true. They may be familiar with the idea that to disprove something you need only show one counter-example and therefore feel that proving should also only involve working with specific cases. However, no matter how many specific cases you've tried, there is always the possibility that there is still one you have not tried that will be a counter-example.

One possible case that you could use to illustrate this point would be to theorize that anytime you square a fraction the result is less than the original fraction. Examples abound, such as  $(1/2)^2 = 1/4$ , and a supporting argument is the oft-heard comment that if you multiply by something less than one you wind up with less than you started with. But what happens with a negative fraction? Since  $(-1/2)^2$  also equals  $1/4$ , the squared result is now greater and the theory fails.

The algebra required to complete this problem is multiplying and simplifying polynomials, though students who are familiar with factoring a difference of squares might choose to apply that concept.

### Answer Check

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After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually *get* the answer), we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

### Our Solution

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In the past, we have only made our solutions available to mentors, as we felt using authentic student work provided better examples of how they actually solve the problem. However, we also realized that sometimes having the solution ahead of time could be helpful, as we often include tips for how to support students in different areas of their work, or in thinking about different parts of the problem. When appropriate, we also include multiple ways to solve the problem.

### Scoring Rubric

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The problem-specific rubric is something we write for every problem for use by those who are assessing student work. It spells out what we expect from students in three areas of problem solving and three areas of communication. The goal is to assess a student response within each category as it relates to the specific criteria for that category. This approach to assessing student work allows you to retrieve more targeted information on the students' areas of strength and weakness.

In the most general sense, Interpretation usually includes understanding the given information, including any diagram, attempting to answer all parts of all of the questions asked, and exhibiting understanding of any fundamental math concepts necessary to solve the problem. Strategy is usually then applying all of that knowledge in a systematic and mathematically sound way that doesn't rely on any lucky guesses.

The samples of student work included in this packet represent a broad range of both writing and problem solving skills. We're focusing on Strategy in this case, so you'll see the score we'd give each student in that category. We also include comments that suggest what might be a good next step for that student to work on in terms of making progress on the problem or improving their solution. Your students might enjoy and benefit from seeing examples of what other students have done once they've completed their own work on the problem.

Many students are in the habit of choosing as many variables as they need to represent the quantities in a problem rather than trying to use given relationships between those quantities to express them in terms of a single variable. For example, if they are told that two numbers sum to 10 they prefer calling the numbers  $x$  and  $y$  and writing  $x + y = 10$  to calling the numbers  $x$  and  $10 - x$ . Often that just means more substituting as they work through the problem.

In this case, though, it led to an erroneous statement. Students chose to call the consecutive integers  $x$  and  $y$ , then represented Alvin's Theorem as  $x^2 - y^2 = x + y$ . Clearly this is not true unless it's accompanied by a second equation stating that  $x = y + 1$  or  $y = x - 1$ , but many students did not include that second constraint. The math is much easier if they choose  $x$  and  $x + 1$  or  $x - 1$  and  $x$  as the two consecutive integers.

Another common mistake was assuming that just showing one more specific example was sufficient to prove Alvin's Theorem. Other students wrote a correct algebraic statement of the theorem, but then "proved" it by substituting a specific value in for  $x$  rather than simplifying.

We're excited about providing these resources to you and hope you find them useful. We'd love to get feedback and ideas from you on the algpow-teachers discussion group.

– RIZ

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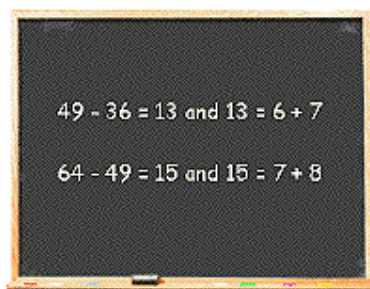
**Problem**

**Alvin's Theorem**

Ms. Powers asked her class to look at a list of square numbers and see if they could find any interesting patterns. After a few minutes, Alvin raised his hand and said, "Ms. Powers, I think I've found something cool. If I take two consecutive squares and subtract them, the difference is always the sum of two consecutive integers."

"Show the class what you mean by that, Alvin," said the teacher.

Alvin wrote the following on the board:



Turning to the class, he said, "I call this 'Alvin's Theorem.'"

Ms. Powers smiled and said, "Very good, but if you want to call it a theorem, you must be able to prove that it's true for every possible pair of consecutive square numbers. Can you do that using algebra?"

Alvin worked for a while and then said, "Yes, I can do that, too. Here's how."

What might Alvin have written on the board next?

**Extra:** Later, Alvin found a similar pattern for the difference of every other square, such as  $49 - 25$  or  $64 - 36$ . What pattern involving a sum do you think he discovered this time?

## Answer Check

One way to solve this problem is to express the difference of consecutive square numbers in terms of a variable, and then show that expression can be simplified to an expression that represents the sum of consecutive integers.

If your answer **doesn't** seem similar to ours, think about these things:

- \* Did you remember to use variables and not just specific examples like the ones Alvin originally wrote on the board?
- \* Did you check your variables and expressions to make sure they accurately represent the information given in the problem?
- \* If you call the first square number  $x^2$ , what would the next square number be?

If any of those ideas help you, you might revise your answer, and then leave a comment that tells us what you did. If you're still stuck, leave a comment that tells us where you think you need help.

If your answer **does** seem similar to ours,

- \* have you clearly shown and explained the work you did?
- \* did you make any mistakes along the way? If so, how did you find and fix them?
- \* are there any hints that you would give another student?
- \* did you try the Extra?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did - you might answer one or more of the questions above.

## Our Solution

Let's start by labeling our two square numbers using a variable:

$$\begin{aligned}x^2 &= \text{smaller or first square number} \\(x + 1)^2 &= \text{larger or second square number}\end{aligned}$$

We're interested in exploring the difference between those square numbers, which can be written as:

$$(x + 1)^2 - (x)^2$$

There are at least two possible solution paths from here. One is to expand the binomial and simplify the resulting expression:

$$\begin{aligned}(x + 1)^2 - (x)^2 \\x^2 + 2x + 1 - x^2 \\2x + 1\end{aligned}$$

Re-writing the last line shows that the difference can in fact be written as the sum of consecutive integers:

$$\begin{aligned}2x + 1 \\x + x + 1 \\(x) + (x + 1)\end{aligned}$$

A second possible solution path is to factor the original difference expression as a difference of squares where  $a^2 - b^2 = (a + b)(a - b)$ :

$$\begin{aligned}(x + 1)^2 - (x)^2 \\(x + 1 + x)(x + 1 - x) \\(x + 1 + x)(1) \\(x + 1 + x) \\(x) + (x + 1)\end{aligned}$$

It's interesting to note that in this case both the consecutive integers at the end and the square numbers at the beginning are expressed by  $(x)$  and  $(x + 1)$ . That may actually prove helpful to some kids as they work on this problem, but it's relatively coincidental - for example, that pattern does not hold in the Extra question.

Some students will notice that the same consecutive integers work on both sides of Alvin's examples, and will choose to approach the problem using that relationship, starting with:

$$(x + 1)^2 - (x)^2 = (x) + (x + 1) \quad \text{or} \quad (x)^2 - (x + 1)^2 = (x - 1) + (x)$$

This approach should be treated as an identity, and students should work to show that the two sides of the equation are identical. They must state that they have shown an identity and conclude that therefore it is true for any value of  $x$ .

**Extra:** Starting the same way as in the problem, but using every other square number:

$$\begin{aligned} x^2 &= \text{smaller or first square number} \\ (x + 2)^2 &= \text{larger or second square number} \end{aligned}$$

Again, the difference expression can be expanded or factored:

$$\begin{array}{l} (x + 2)^2 - (x)^2 \\ x^2 + 4x + 4 - x^2 \\ 4x + 4 \end{array} \qquad \begin{array}{l} (x + 2)^2 - (x)^2 \\ (x + 2 + x)(x + 2 - x) \\ (2x + 2)(2) \\ 4x + 4 \end{array}$$

Keep in mind that the question specifically asked for a relationship involving a sum. Thus, re-writing the last step as  $4(x + 1)$  and stating that the difference is equal to 4 times the number between the two numbers being squared, while correct, is not the desired answer. Our hope is that students will decide that the difference can be written as the sum of consecutive odd integers:

$$\begin{array}{l} 4x + 4 \\ (2x + 1) + (2x + 3) \end{array}$$

Students might find other possible sum expressions, such as doubling the sum of the original numbers:

$$\begin{array}{l} 2[(x) + (x + 2)] \\ 2(2x + 2) \\ 4x + 4 \end{array}$$

If the expression involves a sum, it meets the condition of the problem, but students should be encouraged to revise and think about a nice simple sum involving some sort of consecutive integers.

# Algebra Problem of the Week Scoring Rubric for Alvin's Theorem

For each category, choose the level that *best describes* the student's work.

Problem Solving		Novice	Apprentice	Practitioner	Expert
<b>Interpretation</b>	shows little understanding of the concepts involved - see the Practitioner column	shows understanding of most but not all of the concepts in the Practitioner column  (for example, understands the vocabulary in the problem and tries to prove Alvin's Theorem, but does so by writing more specific examples)	picks an incorrect strategy or relies on luck  doesn't express consecutive integers or squares using variables  hasn't written an expression for the difference of consecutive squares OR has written one but hasn't shown how their expression proves the theorem (i.e. might not simplify it or might use a specific case to try and prove it)	understands what "consecutive squares" and "consecutive integers" mean  understands that proof requires a general case with variable(s), not just more specific examples  attempts to show a proof  success achieved through skill, not luck  uses algebraic techniques to solve the problem – this might include: <ul style="list-style-type: none"> <li>expressing consecutive integers as <math>x</math> and <math>x+1</math> or <math>x-1</math> and <math>x</math></li> <li>writing an expression for the difference of consecutive squares and simplifying</li> <li>writing the theorem as an identity or equation and showing the sides are equal</li> </ul>	solves the main problem and the Extra correctly, and is at least a Practitioner in Strategy  uses two separate strategies or an unusual or sophisticated strategy  (for example, after $(x + 1)^2 - (x)^2$ they might expand and simplify, and also factor as a difference of squares to simplify)
<b>Strategy</b>	has few ideas that will lead them toward a successful solution				
<b>Accuracy</b>	work contains many errors	work is mostly accurate, with a few errors	work is accurate and contains no arithmetic mistakes  uses appropriate vocabulary (integers, squares)	work is accurate and contains no arithmetic mistakes  uses appropriate vocabulary (integers, squares)	generally not possible - can't be more accurate than Practitioner
<b>Communication</b>					
<b>Completeness</b>	has written very little that tells or shows how they found their answer	does not define variable(s)  explains the steps used to find the answer but shows very few of the calculations and work  OR  shows the work but does not explain the thinking behind it	explains the steps used to find the answer but shows very few of the calculations and work  OR  shows the work but does not explain the thinking behind it	defines variable(s)  explains and shows all of the steps taken to solve the problem	adds in useful extensions and further explanation of some of the ideas involved  the additions are helpful, not just "I'll say more to get more credit"
<b>Clarity</b>	explanation is very difficult to read and follow	explanation isn't entirely unclear, but would be hard for another student to follow  explanation is long and is written entirely in one paragraph  explanation contains many spelling and typing errors	explanation isn't entirely unclear, but would be hard for another student to follow  explanation is long and is written entirely in one paragraph  explanation contains many spelling and typing errors	explains all of the steps in such a way that another student would understand them  makes an effort to check their formatting, spelling, and typing (a few errors are fine as long as they don't make it hard to read)	formats things exceptionally clearly  answer is very readable and appealing
<b>Reflection</b>	The items in the columns to the right are considered reflective. They could be in the solution or the comment left after viewing the Math Forum's answer.  did nothing reflective	checked answer in some way (in addition to viewing the answer provided by the Math Forum)  reflected on the reasonableness of their answer  did one reflective thing	checked answer in some way (in addition to viewing the answer provided by the Math Forum)  reflected on the reasonableness of their answer  did one reflective thing	connected the problem to prior problems or experiences  explained where they are stuck  summarized the process they used  did two reflective things	commented on and explained the ease or difficulty of the problem  revised and improved their work  did three or more reflective things or did an exceptional job with two of them

**Jenn**  
age 13  
  
Strategy  
**Novice**

He wrote every square on the board.  
Start with 1 and write every square on the board.

*Short of knowing that the problem involves square numbers, Jenn has written nothing that suggests she is on her way to solving the problem. I'd ask her to tell me what she noticed about Alvin's number examples and not get into any algebra question until the next revision.*

**Daniel**  
age 14  
  
Strategy  
**Novice**

Alvin would write this next:  $81 - 64 = 17$  and  $17 = 8 + 9$   
Alvin would have written this next simply because it works! But, to be little more detailed : This is how he got to his conclusion.  
Now, in the pattern Alvin just subtracted a square minus the square before it ( $64 - 49$ ). So, in the next step to Alvin's theorem he must follow his pattern, but with the next square. In the first step Alvin used 36 and 49, the in the next step he used 49 and 64. It is fair to guess that in the next step Alvin will use 64 and the following square, 81.  
 $81 - 64 = 17$   
This is perfect! The answers to thje first two steps were 13 and 15. If we assume that the next answer will again be two numbers up then the answer to this next one would be 17! Well, now this is perfect, let's see if the next part of this step three works.  
In the first two steps the answer was equal to two consecutive integers. The first being  $6 + 7$ , then  $7 + 8$ . So, it is safe to assume that the next set will be  $8 + 9$ . Now, let us think, what is  $8 + 9$ ?  
Could it be... 17!!!!!!!!!!!!!!!!!!!!  
It is! It is! We now have proven Alvin's Theorem.

*Daniel has noticed a number pattern in Alvin's work and continues the difference of squares, but he has missed the idea that he needs to use algebra to show that Alvin's Theorem will always be true. While that's partly an Interpretation issue, his lack of using a variable or any algebra would make him a Novice in Strategy. I'd start by suggesting that he use  $x$  to represent an integer and asking him what the next consecutive integer after  $x$  would be.*

**Phillip**  
age 15  
  
Strategy  
**Apprentice**

$(x + 1)^2 - x^2 = (x + 1) + x$   
If the first two numbers are consecutive squares, and if you subtract the lesser from the greater, then they will equal the square roots of the two consecutive numbers added together.

*Phillip is off to a good start by using a variable and writing a correct statement of the problem, but has not made any attempt to prove his statement is true. I'd commend him on his expression and ask him to work on simplifying the left side and showing that it will always equal the right side. He should also explain how he chose  $x$  and  $x + 1$ , but that's a Completeness issue and not Strategy.*

**Paul**  
age 14  
Strategy  
**Apprentice**

If I take two consecutive squares and subtract them, the difference is always the sum of two consecutive integers.

I think that Alvin would have written the following on the board:

$x$  is equal to any number.

$$(x+1)^2 - x^2 = x + (x+1)$$

if  $X=1$

$$(10+1)^2 - 10^2 = 10 + (10+1)$$

$$121 - 100 = 10 + 11$$

$$21 = 21$$

*Like Phillip above, Paul is also on the right track with his variable and algebraic expression, but his proof consists of trying one specific value of  $x$ . I'd point out that no matter how many specific cases he tries, he can never be sure it will always be true and ask him to work on showing that the left side of his equation can be simplified to equal the right side.*

**Jorge**  
age 16  
Strategy  
**Practitioner**

$$(a + 1)^2 - a^2 = a + (a + 1)$$

|                    $\wedge$ -----:the sum of two consecutive integers

$\wedge$ -----:the difference of two consecutive squares

---So now we prove that  $a = a$ ---

$$(a + 1)^2 - a^2 = a + (a + 1) \quad \text{:add } a^2 \text{ to both sides and simplify}$$

$$(a + 1)^2 = a^2 + 2a + 1 \quad \text{:now just factor a bit}$$

$$(a + 1)^2 = (a + 1) * (a + 1) \quad \text{:simplify and take the square root}$$

$$a + 1 = a + 1 \quad \text{:subtract one}$$

$$a = a \quad \text{: and you're done :)}$$

*Jorge has done a lot of good algebra, and clearly understands what he is trying to show, which I would tell him. His approach is an "identity" where he has started with an equality statement and then tries to show that it is in fact always equal. Often with identities the expectation is that one side only is simplified to show that it's equal to the other side. I would not hold that subtlety against Jorge, so he's a Practitioner in Strategy, but I'd explain it to him and challenge him to try working with only one side. He's clearly strong in algebra, so I'm confident he'll be able to.*

**Akhil**  
age 15  
Strategy  
**Practitioner**

Alvin's Theorem's proof by simple algebra and induction

Let  $x$  and  $(x + 1)$  be the two consecutive integers Alvin is talking about. Then subtracting the square of the larger number by the smaller we get,

$$(x + 1)^2 - x^2 \quad \text{-----} > (1)$$

On factorizing (1) we get:

$$(x + 1 + x)(x + 1 - x)$$

$$= (x + 1 + x)(1)$$

$$= x + 1 + x$$

$$= x + (x + 1)$$

i.e. the sum of the initial numbers we took!!

*Akhil's algebra is right on the money. I'd ask him to explain a little more about how he did his factoring since the difference of squares formula may not be obvious or known to some students reading his work. I'd also challenge him to see if he could find another way to simplify (1) and wind up with the same final result.*

**Jane**  
age 14  
Strategy  
**Practitioner**

Answer:  $(x + 1)^2 - x^2 = 2x + 1$  and  $2x + 1 = x + (x + 1)$

Let  $x$  be the number of the lesser number. If  $x$  is the lesser number, then next number will be  $x+1$ .

$$x \quad x + 1$$

Because the questions says two consecutive "square" number, so both  $x$  and  $x+1$  is squared.

$$(x+1)^2 \quad x^2$$

There is a minus sign between the two consecutive squares since the question wants you to find the difference between them.

$$(x+1)^2 - x^2$$

I know that  $(x+1)^2$  equal  $(x+1)(x+1)$ , so I will replace  $(x+1)(x+1)$  for  $(x+1)^2$ .

$$(x+1)(x+1) - x^2$$

Multiply  $(x+1)$  and  $(x+1)$

$$x^2 + 2x + 1 - x^2$$

$$= 2x + 1$$

$$= x + (x + 1)$$

$x + (x + 1)$  represents the sum of two consecutive integers. So this proves that "Alvin's Theorem" is true for every possible pair of consecutive square numbers.

*Jane has done a very nice job of proving Alvin's Theorem using a general case. She's established the consecutive square numbers, squared the binomial, and simplified the result into a form that is easily identifiable as the sum of consecutive integers. I'd commend Jane and ask her if she can think of another algebraic way to prove the theorem. Jane would also receive high marks in Completeness and Clarity as her solution is very nicely presented. I'd also ask her to try the Extra question, which is more open-ended.*

**Johan**  
age 15  
Strategy  
**Expert**

For every possible pair of consecutive squares, the difference is ALWAYS the sum of two consecutive integers.

Well, this "Alvin's Theorem" could be explained with a simple algebraic equation. Let's start.

Alvin said that If he took two consecutive squares and subtract them, the difference is always the sum of two consecutive integers. So, let the 1st number be "n" and the 2nd one (it's consecutive integer) is "n+1".

The difference of two consecutive squares can be expressed then as

$$(n+1)^2 - (n)^2$$

From any basic algebra books, we can find this formula:

$$(a+b)^2 = a^2 + 2*a*b + b^2$$

"a" is analogous to "n" and "b" is similar to "1". This will make:

$$(n+1)^2 = n^2 + 2*n*1 + 1^2$$

$$= n^2 + 2*n + 1$$

Then, our equation will become:

$$(n+1)^2 - n^2 = (n^2 + 2*n + 1) - n^2$$

$$= 2*n + 1$$

$$= n + n + 1$$

$$= n + (n+1)$$

$$= (n+1) + n$$

*Johan's strategy is excellent. He establishes the consecutive integers and their squares, then offers two ways to show that Alvin's Theorem will always be true for any possible integer n. He's also done a nice job of presenting his thinking and work in a clear and complete fashion. There's not much to say to Johan other than asking him to take a shot at the Extra question.*

There's another way to prove this "Alvin's Theorem". We know the formula of subtraction for square variables is :

$$a^2 - b^2 = (a+b) * (a-b)$$

Let  $a = n+1$  and  $b = n$ . Substitute  $(n+1)$  and  $n$  to the formula:

$$\begin{aligned}(n+1)^2 - n^2 &= ((n+1) + n) * ((n+1) - n) \\ &= (2*n+1) * 1 \\ &= 2*n + 1 \\ &= n + n + 1 \\ &= n + (n+1) \\ &= (n+1) + n\end{aligned}$$

Algebraic conclusion:  $(n+1)^2 - n^2 = (n+1) + n$

Note: Strictly you can use "n-1" as a substitute for "n" and "n" as substitute for "n+1". It won't make any changes to final solution.