Ceci n'est pas un headnote.

1 *Sum-mer Vacation*

Hey, welcome to the class. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How did others at your table think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Learn from others.** Give everyone the chance to discover, and look to your tablemates for new perspectives on problems. Resist the temptation to tell others the answers if they aren't ready to hear them yet. If you think it's a good time to teach your tablemates about Dirichlet characters, think again: the problems should lead to the appropriate mathematics rather than requiring it. The same goes for technology: the problems should lead to the appropriate use of technology rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff, and maybe other stuff sometimes. Check out Important Stuff first. All the mathematics that is central to the course can be found and developed in the Important Stuff. *That's* why it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we probably noticed... and that question will be seen again soon. Each problem set is based on what happened before, either in problems or in class discussions.

At least one problem in an upcoming problem set is unsolved. You should solve it!

On Day 3, go back and read these again.

Will you remember? Maybe!

Important Stuff

Every positive integer has divisors, numbers that divide evenly into it. The divisors of 4 are 1, 2, and 4. The divisors of 18 are 1, 2, 3, 9, and 18, and maybe one more.

Make sure you bring divisors if you're going to da beach and da sun is out.

We're live, investigating a function called σ, which takes in a positive integer, and spits back the *sum* of all its divisors. For example, $\sigma(4) = 7$ and $\sigma(18) \geq 33$.

PROBLEM

Here's a massive table for the σ function. Complete the table without using any technology developed after 1565.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\sigma(n)$				7								
n	13	14	15	16	17	18	19	20	21	22	23	24
$\sigma(n)$												
n	25	26	27	28	29	30	31	32	33	34	35	36
$\sigma(n)$												
n	37	38	39	40	41	42	43	44	45	46	47	48
$\sigma(n)$												
n	49	50	51	52	53	54	55	56	57	58	59	60
$\sigma(n)$					54							
n	61	62	63	64	65	66	67	68	69	70	71	72
$\sigma(n)$												
n	73	74	75	76	77	78	79	80	81	82	83	84
$\sigma(n)$												
n	85	86	87	88	89	90	91	92	93	94	95	96
$\sigma(n)$												

Stuff in boxes is more important than other Important Stuff! By the way, the pencil was invented around 1560.

1. Describe some patterns in the table for the σ function, especially patterns that helped you complete the table quickly, or patterns you could use to find other outputs.

2. Determine each of the following. Hey, stop trying to use a calculator!
(a) $\sigma(128)$
(b) $\sigma(243)$
(c) $\sigma(5 \cdot 49)$
(d) $\sigma(257)$
(e) $\sigma(1001)$

5 times 49? You're making it way more complic. . . Ohhhhh. Cool.

3. Define $A(n) = \frac{\sigma(n)}{n}$. Alright, fine, you can start using a calculator now.
(a) Find all numbers n with $A(n) \leq 1$.
(b) Find three numbers n with $A(n) = 2$.
(c) Are there any numbers n with $A(n) = 3$?

Neat Stuff

Here are some more good questions to think about.

4. If p is prime, what can you say about $A(p)$?

I heard $A(p)$ is a freeloader.

5. If p and q are primes, what can you say about $A(pq)$?

6. If p and q are primes, find the maximum possible value of $A(pq)$.

7. Without relying on technological crutches, find a number for which $\sigma(n) = 1000$ or show that no such number exists.

8. Find the maximum possible value of $A(n)$.

9. Go back to the problems on page 2, except now use $\sigma_2(n)$, the sum of the *squares* of the divisors, and $B(n) = \frac{\sigma_2(n)}{n^2}$.

Tough Stuff

Here are four much more difficult problems to try.

10. Find all numbers for which $\sigma(n) = 1000$.

11. Find a number n for which $A(n) \geq 10$, or prove that no such number exists.

12. Find an odd number for which $A(n) = 2$, or prove that no such number exists.

13. Find the maximum possible value of $B(n)$.

It's on the number line somewhere, so how hard can it be to find, right? Right?

Ceci n'est pas un headnote.

This page intentionally left blank.

Watch this space for important messages. Eventually there might be one...

2 *Sum-Thing To Talk About*

Important Stuff

As you learned yesterday, the sum of the divisors of a number is cool. So, the sum of the divisors' *reciprocals* has to be ice cold! Today Todd, Todd, and the rest of our investigatory team tackle a function called a, which takes in a positive integer, and spews forth the sum of its divisors' reciprocals. For example, $a(12) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{7}{3}$.

This is known as André 3000's Theorem, but he's a bit of a mathematical outkast.

PROBLEM

Here's a table for the a function. Complete the table without using anything that can calculate faster than you. Write answers in "lowest terms."

n	1	2	3	4	5	6	7	8	9	10	11	12
$a(n)$												$\frac{7}{3}$

n	13	14	15	16	17	18	19	20	21	22	23	24
$a(n)$												

n	25	26	27	28	29	30	31	32	33	34	35	36
$a(n)$												

n	37	38	39	40	41	42	43	44	45	46	47	48
$a(n)$												

See, you still only have to write 96 numbers, since there are 48 numerators and 48 denominators. Aren't we nice? Hey, it wasn't even 96 numbers!

pssst... there's more fun to be had on the next page!

Watch this space for important messages. Eventually there might be one...

1. Determine each of the following.

(a) $a(3) \cdot a(4)$

(b) $a(2) \cdot a(5)$

(c) $a(8) \cdot a(15)$

(d) $a(120)$

(e) $a(10) \cdot a(12)$

Um, shouldn't this be problem #2? Yeah yeah, sure sure.

2. Determine each of the following. No calculator please!

(a) $\left(1+\frac{1}{3}\right)\left(1+\frac{1}{2}+\frac{1}{4}\right)$

(b) $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{5}\right)$

(c) $\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{15}\right)$

(d) $\left(1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}\right)$

3. Calculate each of the following.

(a) $1+\frac{1}{2}$

(b) $1+\frac{1}{2}+\frac{1}{4}$

(c) $\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}$

(d) $\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\cdots\frac{1}{2^6}$

(e) The sum of all numbers in the form $\frac{1}{2^n}$ as n goes from 0 to 10

(f) $\displaystyle\sum_{n=0}^{11}\frac{1}{2^n}$

(g) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^n}$

Zack points out that because it says "calculate", you can use a calculator now!

Sigma is your friend. Or at least, it wants to be your friend. Σ is σ's daddy.

4. Let n be a power of 3. Find the *smallest* possible number k for which you are completely sure that $k > a(n)$.

Neat Stuff

5. For certain values of n, it turns out that $\sigma(n) = 3 + \frac{n}{2} + n$. Classify these numbers and find a generalization.

Remember, the σ function is the sum of the divisors. Or is it? No, it is.

6. If p and q are primes, write a rule for $\sigma(pq)$ in terms of p and q.

7. If p and q are primes, write a rule for $a(pq)$ in terms of p and q and give the simplest answer you can.

8. Let n be a number whose factors consist only of 2s and 3s.

(a) Find the smallest possible number k for which you are completely sure that $k > a(n)$.

(b) Find a suitable number n such that $k - a(n) < 0.1$.

9. Hillary challenges you to find a number for which $\sigma(n) = 1000$ or show that no such number exists.

Please don't disappoint Hillary by using technology to answer this question.

10. Find the maximum possible value of $a(n)$.

11. Find the first ten numerators in the following bizarre-looking expansion. Do *not* try to simplify or combine terms, just expand!

$$\left(\frac{1}{1^x}+\frac{2}{2^x}+\frac{3}{3^x}+\frac{4}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

12. Define $\sigma_2(n)$ to be the sum of the squares of the divisors of n and $b(n)$ as the sum of the reciprocals of the squares of the divisors of n.

(a) Tabulate the σ_2 function from 1 to 10.

(b) Find some interesting things about the σ_2 function.

(c) Calculate $\sigma_2(120)$ as easily as possible. Without a calculator please.

(d) Find a number with $b(n) = 2$, or prove that no such number exists.

That's way too many "of the"s for a reasonable sentence. Sure would be nice to have some notation to simplify these kinds of descriptions. Of the.

13. Find the first ten numerators in the following bazaar-looking expansion. Do *not* try to simplify or combine terms, just expand!

$$\left(\frac{1}{1^x}+\frac{4}{2^x}+\frac{9}{3^x}+\frac{16}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

Tough Stuff

14. Find all numbers for which $\sigma(n) = 1000$.

15. Find a number n for which $a(n) \geq 5$, or prove that no such number exists.

16. Find an odd number for which $a(n) = 2$, or prove that no such number exists.

17. Find the maximum possible value of $b(n)$.

This page intentionally left blank.

Jokes? *Jokes?* We don't need no stinkin' jokes!

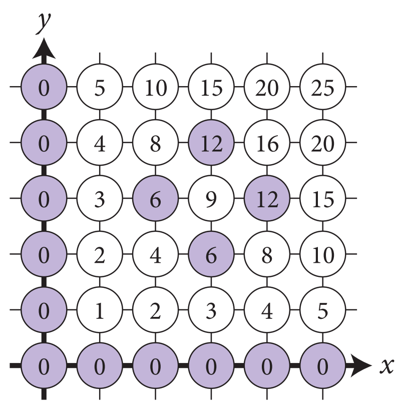
3 *Dim Sum*

Important Stuff

The figure below tabulates the product xy for different values of x and y. How many of these numbers are multiples of 6?

Today, Kim, Kym, and our i-team investigates $k(n)$, which takes in an integer and expectorates the number of products xy that are multiples of n for x and y ranging from 0 to $n-1$.

Err... Did you notice that the multiples of 6 are shaded in the figure below? Yep, 0 is a multiple of 6.



Based on the picture above, $k(6) = 15$.

PROBLEM

Here's a table for the k function. Complete the table without using anything containing silicon. Use the handout to help you.

n	1	2	3	4	5	6	7	8	9	10	11	12
$k(n)$	1					15						

n	13	14	15	16	17	18	19	20	21	22	23	24
$k(n)$												

1. Notice anything interesting about $k(n)$?

Sorry, but "Nope" is not an acceptable answer here.

Jokes? *Jokes?* We don't need no stinkin' jokes!

2. (a) Armando says that if a number is a multiple of 8 and a multiple of 15, then it must be a multiple of 120. What do you think?

(b) Jennifer says that if a number is a multiple of 10 and a multiple of 12, then it must be a multiple of 120. What do you think?

Which Jennifer? We're not telling.

3. Use the units digit of each number to determine which of the following multiplications was performed *incorrectly.*

(a) $234 \times 153 = 35802$

(b) $157 \times 321 = 50397$

(c) $223 \times 155 = 34565$

(d) $168 \times 183 = 30746$

4. When working with units digits you are working in "mod 10", a number system that considers only remainders when dividing by 10. In mod 10, the only numbers are 0 through 9. There are lots of other "mods" too.

If all these mods got together to form a team, it would be the Mod ... Cabal? Something like that.

(a) What is 3×4 in mod 10?

(b) What is 8×3 in mod 10?

(c) What is 8×3 in mod 12?

(d) What number x solves $2x = 7$ in mod 9?

(e) What *two* numbers x solve $x^2 = 7$ in mod 9?

The answer isn't $\sqrt{7}$ here! The only numbers are 0 through 8. But the answers do "act like" $\sqrt{7}$ in some way.

5. Consider this summation:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}}_{\text{8 of these}} + \cdots$$

where each term is repeated 1, then 2, then 4, then 8 times, and it goes on forever. What happens to the sum as you take more terms? Is there a limit to the maximum value of the sum?

This problem was featured in the Johnny Lott biopic "Walk the Number Line."

6. A function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't share any common factors higher than 1.

Felipe calls this situation "relatively prime," which is the more technical term for not sharing common factors. So if you hear that, it means the other thing.

(a) Give three examples of multiplicative functions you've seen in this course.

(b) Give three more examples of multiplicative functions.

Did you re-read the first page from Day 1 like we said you should? Good for you!

Jokes? *Jokes?* We don't need no stinkin' jokes!

Neat Stuff

7. Demonstrate each of the following, any way you like.

(a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$

(b) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{3}{2}$

(c) $\frac{1}{4^0}+\frac{1}{4^1}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{4}{3}$

(d) $\sum_{n=0}^{\infty}\frac{1}{5^n}=\frac{5}{4}$

(e) $\sum_{n=0}^{\infty}\frac{1}{p^n}=\frac{p}{p-1}$

This last bit isn't actually true all the time! Feel free to investigate but we will only be working with numbers that make this true.

8. Remember $a(n)$ from yesterday? Sure you do! Consider $a(72)$.

(a) What is the value of $a(72)$?

(b) Calculate

$$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)\left(1+\frac{1}{3}+\frac{1}{9}\right)$$

(c) *Without performing the final addition*, expand the expression above. What happens?

One of the terms in the expansion is $\frac{1}{18}$.

9. The function $\tau(n)$ is defined as the number of factors of n.

(a) Tabulate the τ function for $n = 1$ through 20.

(b) Is τ multiplicative? Can you prove it?

(c) Describe a way to calculate $\tau(n)$ for any integer n.

10. Let $n = 2^p3^q$. Find some values of p and q that produce particularly large values of $a(n)$, then determine the maximum possible value of $a(n)$ for any number in this form.

11. Let $n = 2^p3^q4^r$. Determine the maximum possible value of $a(n)$ for any number in this form.

12. Let $n = 2^p3^q5^r$. Determine the maximum possible value of $a(n)$ for any number in this form.

13. Consider this summation:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\cdots$$

What happens to the sum as you take more terms? Is there a limit to the maximum value of the sum?

14. Prove that for any n, $a(n!)$ is larger than the sum of the first n terms in the series of Problem 13.

You may find something helpful in a previous problem, and the "number line jumping" picture may also be useful.

15. Prove that there is no maximum value of $a(n)$.

16. Find the first ten numerators in each of these bizah-looking expansions. Or more than 10, we don't care. Notice anything?

For example, when you end up seeing a term like $\frac{1}{2^x 3^x}$, that's really $\frac{1}{6^x}$. But don't try to simplify something like $\frac{2}{2^x}$ to $\frac{1}{2^{x-1}}$, just leave it so the denominators are all k^x.

(a)

$$\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

(b)

$$\left(\frac{1}{1^x}+\frac{2}{2^x}+\frac{3}{3^x}+\frac{4}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

(c)

$$\left(\frac{1}{1^x}+\frac{1/2}{2^x}+\frac{1/3}{3^x}+\frac{1/4}{4^x}+\cdots\right)\left(\frac{1}{1^x}+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots\right)$$
$$=\frac{?}{1^x}+\frac{?}{2^x}+\frac{?}{3^x}+\frac{?}{4^x}+\cdots$$

17. Let $f(n)$ be the number of solutions to $x^2 = 7$ in mod n. Figure out anything you can about this function.

If $n < 7$ you will need to adjust the equation to suit the mod. For example, in mod 5, the equation becomes $x^2 = 2$.

18. Define $\sigma_2(n)$ to be the sum of the squares of the divisors of n and $b(n)$ as the sum of the reciprocals of the squares of the divisors of n.

(a) Tabulate the σ_2 function from 1 to 10.

(b) Find some interesting things about the σ_2 function.

(c) Find $\sigma_2(120)$ without a calculator.

Tough Stuff

19. Find a number n for which you can prove *without use of any technology* that $a(n) > 10$.

One such number was found yesterday, by brute force calculation—it was over 200 digits long. Probably the one you find here will be much longer.

20. For what primes p is 7 a perfect square in mod p?

21. Find the maximum possible value of $b(n)$, where b is the function from Problem 18.

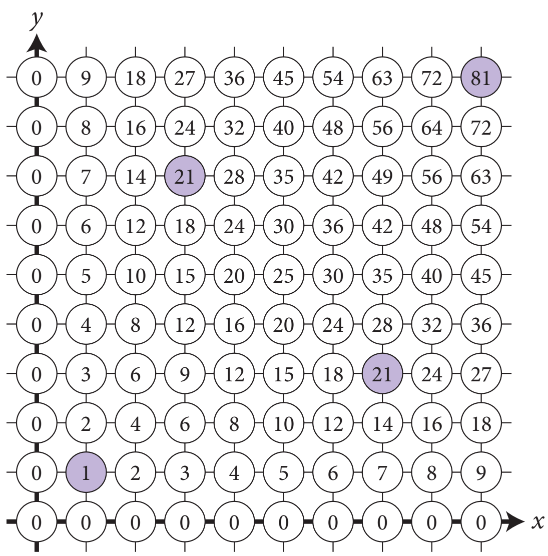
Congrats to Bill on his successful singing of James Brown's "Get On Up"!

4 *Sum-thing Wicked This Way Comes*

Important Stuff

The figure below tabulates the product xy for values of x and y from 0 to 9. How many of these products end in 1? All these numbers are 1 more than a multiple of 10. Today Emily and our mod squad investigate $P(n)$, which takes in an integer and regurgitates the number of products xy that are *one more* than a multiple of n. Like yesterday, x and y range from 0 to $n - 1$.

So, you can use the same table as yesterday, but we've provided lots of copies with that annoying diagonal filled in too!



Based on the picture above, $P(10) = 4$, since there are four numbers that are 1 more than a multiple of 10.

Remember, for $P(9)$ you'd want to find numbers that are 1 more than a multiple of 9.

PROBLEM

Here's a table for the P function. Complete the table without using anything that has an "enter" key. Use the handout to help you, and don't forget to cheat appropriately.

n	1	2	3	4	5	6	7	8	9	10	11	12
$P(n)$	1		2					4		4		

n	13	14	15	16	17	18	19	20	21	22	23	24
$P(n)$												

1. Notice anything interesting about $P(n)$?

It's okay to say "nah," but then you still gotta find sum-thing.

2. Describe how you could directly find $P(n)$ for any n, then use your method to find $P(210)$.

3. A function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ whenever a and b don't share any common factors higher than 1.

If a and b share common factors, all bets are off!

(a) Give *four* examples of multiplicative functions you've seen in this course.
(b) Give one more example of a multiplicative function.
(c) Give three examples of functions of functions that are *not* multiplicative.

4. (a) Katie says that if a number is one more than a multiple of 8 and one more than a multiple of 15, then it must be one more than a multiple of 120. What do you think?

A déjà vu is usually a glitch in some matrices. It happens when Bowen and Darryl change something.

(b) Cliff says that if a number is one more than a multiple of 10 and one more than a multiple of 12, then it must be one more than a multiple of 120. What do you think?

5. Functions can have babies! Define the *child* g of a function f by the following:

$g(\text{Aaron}) = f(\text{Bowen}) + f(\text{Nancy})$

$$
\begin{aligned}
g(n) &= f(\text{all divisors of } n) \text{ added together} \\
g(15) &= f(1) + f(3) + f(5) + f(15) \\
g(20) &= f(1) + f(2) + f(4) + f(5) + f(10) + f(20) \\
g(1) &= f(1)
\end{aligned}
$$

Let $r(n) = n$, and let s be the child of r.
(a) Calculate $s(1)$ through $s(10)$, $s(15)$, and $s(20)$.
(b) Is r multiplicative? Does s seem to be multiplicative?
(c) Hey, where have we seen s before?

6. Find all solutions to these equations. Reminder: in mod 7, the only possible answers are 0, 1, 2, 3, 4, 5, and 6.

Just like there is more than one John in the room, there can be more than one answer to each question. Or no answer at all.

(a) $3x = 4$ in mod 7
(b) $6x = 4$ in mod 7
(c) $6x = 4$ in mod 8
(d) $6x = 1$ in mod 8
(e) $x^2 = 2$ in mod 7
(f) $x^3 = 1$ in mod 7
(g) $x^3 = -1$ in mod 7
(h) $x^6 = 1$ in mod 7

Wait, -1 doesn't exist in mod 7...oh right, it's 6.

Congrats to Bill on his successful singing of James Brown's "Get On Up"!

Neat Stuff

7. Many pairs of numbers have no common factor higher than 1: for example, 8 and 15. Function $\phi(n)$ returns the number of values from 1 to n that, when checked against n, have no common factor higher than 1.

(a) Show that $\phi(3) = 2$, $\phi(5) = 4$, and $\phi(15) = 8$.

(b) Calculate values of the ϕ function until you figure out what is happening.

How do *you* pronounce that greek letter ϕ? Is it "phee, phi, pho, phum" or "phi, phy, pho, phum"? Only your hairdresser knows for sure!

8. Let $S = 1 + 5 + 5^2 + 5^3 + \cdots + 5^n$.

(a) Write an expression for $5S$.

(b) Write a really clever expression for $4S$ by subtracting.

(c) Show that

$$S = \frac{5^{n+1} - 1}{4}$$

(d) Find a general rule for $1 + r + r^2 + r^3 + \cdots + r^n$.

9. What is the child of $f(n) = 1$?

10. Remember function k from yesterday? Find the child of k and tabulate its values. What patterns do you observe?

Sometimes children are easier to understand than parents...

11. Prove that the sum of the harmonic sequence

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$$

does not converge to any real number. Bonus points if you use a method different from the one presented today.

12. Remember function a? Write out the largest eight fractions that are part of each of these.

(a) $a(24)$

(b) $a(720)$

(c) $a(10!)$... make it the largest twelve fractions here.

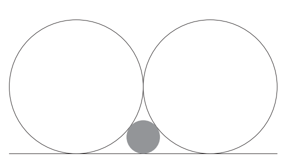
13. Prove that for any n, $a(n!)$ is larger than the sum of the first n terms in the series of Problem 11.

Pronounce this function as "a of $ENNN$!"

14. Prove that there is no maximum value of $a(n)$.

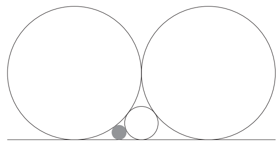
15. Two circles with diameter 1 are tangent to a line, as well as to each other. What is the diameter of the gray circle that is tangent to both circles as well as the line?

And now, for something completely different! *Or is it?* No, it is.



Have an awe-sum weekend!

16. Start with the diagram from the previous problem, and add one more circle in the hole in the lower left; it is also tangent to the line and two other circles. What is the diameter of this circle (marked in gray below)?



17. (a) Find an equation that has *every number* as a solution in mod 7.

(b) Multiply this out and simplify it in mod 7:

$$x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$$

You might want to use a CAS calculator for this. Get a new calculator window using the HOME icon in the upper right, then select "expand" from the menus. "Simplify in mod 7" means that if you see 9x, write 2x.

18. Let $f(n)$ be the number of solutions to $xy = 5$ in mod n. Figure out anything you can about this function.

19. Hey, go work on that sum of the squares of divisors problem from earlier.

Tough Stuff

20. Prove that the P and ϕ functions must be identical. Specifically, prove that if a is relatively prime to n, then there is exactly one solution to $ab = 1 \bmod n$, and if a isn't relatively prime, there are no solutions.

21. For what primes p is 5 a perfect square in mod p?

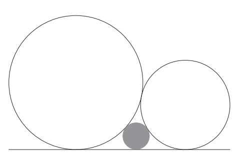
22. What is the *grandparent* of $f(n) = 1$?

Eww, we didn't tell you how to find a parent if you're given the child... and you have to do that twice!

23. $b(n)$ is the sum of the squares of the reciprocals of the divisors of n. Find the maximum possible value of $b(n)$ or prove there is no maximum.

24. Two circles with diameters a and b are tangent to a line, as well as to each other. How does the diameter c of the gray circle compare? Come up with an incredibly nice relationship between a, b, and c.

Is it symmetric in a, b, c? Better be or it's not nice enough!



Have an awe-sum weekend!

5 *Welcome to Sum-it County*

Important Stuff

1. The ϕ function gives the number of values from 1 to n that are relatively prime to n; that is, those number that share no common factors greater than 1 with n. For example, $\phi(6) = 2$ since 1 and 5 don't share common factors with 6 (besides 1), while 2, 3, 4 do.

(a) Determine $\phi(3)$, $\phi(5)$, and $\phi(15)$.

(b) Determine $\phi(2)$, $\phi(7)$, and $\phi(14)$.

(c) If p is a prime number, what is a formula for $\phi(p)$?

(d) Determine a formula for $\phi(pq)$ where p and q are distinct primes.

Here's a handy-dandy table for $\phi(n)$.

Oh it will be useful. . . soon! All too soon.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(n)$	1			2		2		4	6	4	10	4
n	13	14	15	16	17	18	19	20	21	22	23	24
$\phi(n)$	12			8	16	6	18	8	12	10	22	8

2. Let $f(x) = \dfrac{1}{x}$. The *child* g of f is found by calculating

$$\begin{aligned} g(n) &= f(\text{all divisors of } n) \text{ added together} \\ g(15) &= f(1) + f(3) + f(5) + f(15) \\ g(20) &= f(1) + f(2) + f(4) + f(5) + f(10) + f(20) \\ g(1) &= f(1) \end{aligned}$$

(a) Determine $g(3)$, $g(5)$, and $g(15)$.

(b) Determine $g(2)$, $g(7)$, and $g(14)$.

(c) If p is a prime number, what is a formula for $g(p)$?

Thanks for all your hard work and insight this week! We hope you've had a good time and learned a few things. . . more to come.

Have an awe-sum weekend!

PROBLEM

Let r be the child of ϕ, which we defined in Problem 1. Remember, if r is the child of ϕ it means that $r(3) = \phi(1) + \phi(3)$ and $r(10) = \phi(1) + \phi(2) + \phi(5) + \phi(10)$, among others.

Complete this table for $r(n)$ from 1 to 24, and determine a rule for $r(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$r(n)$												

n	13	14	15	16	17	18	19	20	21	22	23	24
$r(n)$												

Cool, huh?

Too soon?

3. Determine the grandchild of the ϕ function by finding the child of r.

4. Find all solutions for each of the following.
(a) $x^2 = x$ in mod 3
(b) $x^2 = x$ in mod 5
(c) $x^2 = x$ in mod 7
(d) $x^2 = x$ in mod 15
(e) $x^2 = x$ in mod 21
(f) $x^2 = x$ in mod 35
(g) $x^2 = x$ in mod 105

Be careful on the fourth!

5. List five things you learned this week.

Mathematical things, not facts like "Oscar's favorite color is puce."

Neat Stuff

6. What is the child of $f(n) = 1$, a constant function?

7. Which of the following functions are multiplicative?
(a) $f(n) = n^2$
(b) $g(n) = 2n$
(c) $m(n) = n \bmod 12$
(d) $a(n) = \frac{\sigma(n)}{n}$
(e) $y(n) = \gcd(n, 12)$

8. If f is a multiplicative function, what are the possible values of $f(1)$?

Have an awe-sum weekend!

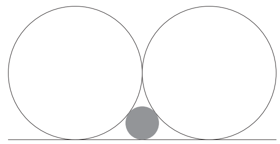
9. Is the sum of two multiplicative functions also multiplicative? What about the product?

While no proof is expected here, you might think about how you would prove some of these. How are multiplicative functions "built"?

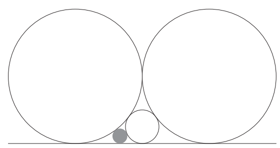
10. Is the child of a multiplicative function also multiplicative?

11. Two circles with diameter 1 are tangent to a line, as well as to each other. What is the diameter of the gray circle that is tangent to both circles as well as the line?

What is this really a picture of? Someone smoking while looking through binoculars? An orange stuck in the cantaloupe aisle?



12. Start with the diagram from the previous problem, and add one more circle in the hole in the lower left; it is also tangent to the line and two other circles. What is the diameter of this circle (marked in gray below)?



13. The function $k_5(n)$ counts the number of solutions to the equation $xy = 5 \bmod n$. Here's a table of some values of $k_5(n)$.

Wouldn't you really rather do this by making a giant chart again? Really? You sure?

n	1	2	3	4	5	6	7	8	9	10	11	12
$k_5(n)$	1	1	2	2	9	2	6	4	6	9	10	4

n	13	14	15	16	17	18	19	20	21	22	23	24
$k_5(n)$	12	6	18	8	16	6	18	18	12	10	22	8

Find a rule for the *child* of k_5.

14. Prove that if a and n are relatively prime, then there is a solution to the equation $ab = k$ in mod n for every number k from 0 to $n - 1$.

Eat Pizza, Do Math: One method uses a pigeonhole argument. What would happen if one of the equations couldn't be solved? Use that to find a contradiction.

15. Find all solutions to each of the following equations.

(a) $x^2 = 1$ in mod 3
(k) $x^4 = 1$ in mod 5
(e) $x^5 = 1$ in mod 7
(m) $x^{10} = 1$ in mod 11
(i) $x^5 = 1$ in mod 6

16. If $x^2 = 1$ in mod p for prime p, what are the possible values of x?

Have an awe-sum weekend!

17. In mod 11, the perfect squares are 1, 4, 9, 5, and 3. Find the perfect squares in each of these mods.
(a) mod 7
(b) mod 13
(c) mod 17
(d) mod 19

And the perfect strangers are Larry and Balki.

18. Find all solutions to each of the following equations.
(a) $x^3 = 1$ in mod 7
(b) $x^3 = -1$ in mod 7
(c) $x^6 = 1$ in mod 13
(d) $x^8 = 1$ in mod 17
(e) $x^9 = 1$ in mod 19

Ooh, a scavenger hunt question.

19. Without technology, find all the solutions to the equation $x^2 - 4x + 3 = 0$ in mod 165.

Aretha says you better think about what you're trying to do to this before brute-forcing it.

20. Let $c(n)$ be the number of solutions to the equation $x^2 = 1$ in mod n. Prove that c is multiplicative.

Tough Stuff

21. Determine the grandparent of $f(n) = 1$.

22. Problem 13 introduced a function called $k_5(n)$ and asked you to find a rule for its child. Find a more general rule for the child of $k_a(n)$, the number of solutions to the equation $xy = a$ in mod n. This rule should work for $a = 0$ and $a = 1$, among others.

23. Let $S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ be the sum of the reciprocals of all odd numbers. Determine, with proof, whether this sum converges or diverges.

24. Let $S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ be the sum of the reciprocals of all prime numbers. Determine, with proof, whether this sum converges or diverges.

25. Let $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{12} + \frac{1}{14} + \cdots$ be the sum of the reciprocals of all numbers *that don't have a 3 in them*. Determine, with proof, whether this sum converges or diverges.

26. Find an odd number such that $\sigma(n) = 2n$, or prove that no such number exists.