More Neat Stuff

Some Counting Problems

1. Find all solutions of each of the following congruences:

a. $x^2 \equiv x \pmod 3$;
b. $x^2 \equiv x \pmod 5$;
c. $x^2 \equiv x \pmod 7$;
d. $x^2 \equiv x \pmod{15}$;
e. $x^2 \equiv x \pmod{105}$;
f. $x^2 \equiv x \pmod{25}$;
g. $x^2 \equiv x \pmod{75}$.

How many solutions are there in each case? Without actually finding all the solutions, can you say $h$ow many solutions there will be modulo 1001? modulo 525? modulo m, where $m = 2 \cdot 3 \cdot 5 \cdot 121 \cdot 17 \cdot 19$?

2. For each of the following find all integers $x \in \mathbb{Z}$ that satisfy both congruences simultaneously:

a. $x \equiv 1 \pmod 3, \quad x \equiv 0 \pmod 5$.
b. $x \equiv 0 \pmod 3, \quad x \equiv 1 \pmod 5$.

3. For each of the following find all integers $x \in \mathbb{Z}$ that satisfy both congruences simultaneously:

a. $x \equiv 2 \pmod 3, \quad x \equiv 0 \pmod 5$.
b. $x \equiv 2 \pmod 3, \quad x \equiv 3 \pmod 5$.

4. For each of the following find all integers $x \in \mathbb{Z}$ that satisfy all three congruences simultaneously:

a. $x \equiv 1 \pmod 3, \quad x \equiv 0 \pmod 5, \quad x \equiv 0 \pmod 7$.
b. $x \equiv 0 \pmod 3, \quad x \equiv 1 \pmod 5, \quad x \equiv 0 \pmod 7$.

c. $x \equiv 0 \pmod 3$, $x \equiv 0 \pmod 5$, $x \equiv 1 \pmod 7$.

5. For each of the following find all integers $x \in \mathbb{Z}$ that satisfy all three congruences simultaneously:

a. $x \equiv 2 \pmod 3$, $x \equiv 0 \pmod 5$, $x \equiv 0 \pmod 7$.
b. $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 0 \pmod 7$.
c. $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 4 \pmod 7$.

6. In each case, how many solutions $x, y \pmod m$ are there of the given congruence for $m = 5, 13, 65, 7, 91$?

a. $x + y \equiv 0 \pmod m$;
b. $x \cdot y \equiv 1 \pmod m$;
c. $x^2 - y^2 \equiv 1 \pmod m$;
d. $x \cdot y \equiv 0 \pmod m$;
e. $x^2 - y^2 \equiv 0 \pmod m$.
f. $x^2 - 4y^2 \equiv 0 \pmod m$.
g. $x^2 + y^2 \equiv 0 \pmod m$.

7. In each case, how many solutions $x, y \pmod m$ are there of the congruence

$$x^2 - y^2 \equiv 0 \pmod m$$

for $m = 3, 5, 7, 11, 13, 17, 19, 23$? How many solutions do you think there will be for $m = 37$? for $m = 59$? for $m = 15$? for $m = 65$?

8. In each case, how many solutions $x, y \pmod m$ are there of the congruence

$$x^2 + y^2 \equiv 0 \pmod m$$

for $m = 3, 5, 7, 11, 13, 17, 19, 23$? How many solutions do you think there will be for $m = 37$? for $m = 59$? for $m = 15$? for $m = 65$?

9. Say whether or not each of the following functions is multiplicative.

a. $f(m) :=$ the number of $x \pmod m$ for which $x^2 \equiv x \pmod m$;
b. $g_1(m) :=$ the number of $x, y \pmod m$ for which $x \cdot y \equiv 1 \pmod m$;
c. $g_2(m) :=$ the number of $x, y \pmod m$ for which $x^2 - y^2 \equiv 1 \pmod m$;

d. $h_1(m)$:= the number of $x, y \pmod{m}$ for which $x^2 - y^2 \equiv 0 \pmod{m}$;
e. $h_2(m)$:= the number of $x, y \pmod{m}$ for which $x^2 + y^2 \equiv 0 \pmod{m}$;

In each case, can you give a formula for the given function?

Miscellaneous problems that continue the themes

10. Suppose g is a multiplicative function and f is g's parent, so that

$$g(n) = \sum_{d|n} f(d)$$

Show that for all primes p and for all positive integers k,

$$f(p^k) = g(p^k) - g(p^{k-1})$$

11. Suppose that $f(n) = n$ and let g be defined by

$$g(n) = \sum_{d|n} \sigma(d) f\left(\frac{n}{d}\right)$$

a. Tabulate g.
b. Is g multiplicative?
c. Investigate g's parent and kid.

12. Suppose that f and h are multiplicative functions. let g be defined by

$$g(n) = \sum_{d|n} f(d) h\left(\frac{n}{d}\right)$$

Show that g is multiplicative.

13. Suppose that u is a multiplicative function so that

$$n = \sum_{d|n} \tau(d) u\left(\frac{n}{d}\right)$$

Find formula for u.

14. Suppose that w is a multiplicative function so that

$$\sum_{d|n} w\left(\frac{n}{d}\right) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

Find formula for w.

15. Let $a(n)$ be defined in this curious way:

$$\begin{aligned} a(n) = &\text{ the number of factors of } n \text{ of the form } 4n+1- \\ &\text{ the number of factors of } n \text{ of the form } 4n-1 \end{aligned}$$

a. Tabulate a.
b. Is a multiplicative?
c. Is $a(n)$ ever negative?
d. Find formulas for a's parent and kid.

16. Let $c(n)$ be defined in this curious way:

$$c(n) = \begin{cases} 1 & \text{if } n = a^2 + b^2 \text{ for integers } a \text{ and } b \\ 0 & \text{otherwise} \end{cases}$$

a. Tabulate c.
b. Is c multiplicative?
c. Find formulas for c's parent and kid.

17. Let $\chi(n)$ is defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv -1 \pmod 4 \\ 0 & \text{if } n \text{ is even} \end{cases}$$

a. Tabulate χ.
b. Is χ multiplicative?
c. Find formulas for χ's parent and kid.

18. Let $s(n)$ is defined by

$s(n) =$ the number of different ways that n can be written as $a^2 + b^2$ for integers a and b

$\longleftarrow$ You decide what "different" means.

a. Tabulate s.
b. Is s multiplicative?
c. Find formulas for s's parent and kid.

19. Let u be the function from problem 13. If f is a multiplicative function and g is the child of f, show that

$$f(n) = \sum_{d|n} g(d) u\left(\frac{n}{d}\right)$$

More Good Stuff

20. More fun calculations:

a. $\left(1+\frac{1}{2^x}\right)\left(1+\frac{1}{3^x}\right)$

b.

$$\left(1+\frac{1}{2^x}+\frac{1}{4^x}\right)\left(1+\frac{1}{3^x}\right)$$

c.

$$\left(1+\frac{1}{2^x}+\frac{1}{4^x}+\frac{1}{8^x}\right)\left(1+\frac{1}{3^x}\right)$$

d.

$$\left(1+\frac{1}{2^x}+\frac{1}{4^x}+\frac{1}{8^x}\right)\left(1+\frac{1}{3^x}+\frac{1}{9^x}\right)$$

e.

$$\left(1+\frac{1}{2^x}+\frac{1}{4^x}+\frac{1}{8^x}\right)\left(1+\frac{1}{3^x}+\frac{1}{9^x}+\frac{1}{27^x}\right)\left(1+\frac{1}{5^x}+\frac{1}{25^x}+\frac{1}{125^x}\right)$$

21. Find closed forms for each infinite sum:

a.

$$\sum_{k=1}^{\infty}\frac{1}{\left(2^x\right)^k}=1+\frac{1}{2^x}+\frac{1}{4^x}+\frac{1}{8^x}+\dots$$

b.

$$\sum_{k=1}^{\infty}\frac{1}{\left(3^x\right)^k}=1+\frac{1}{3^x}+\frac{1}{9^x}+\frac{1}{27^x}+\dots$$

The **Riemann zeta function** $\zeta(s)$ is the formal series:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\dots+\frac{1}{75^s}+\dots$$

$\longleftarrow$ Why s instead of x? Riemann used s, and the convention stuck.

22. Show that

$$\begin{aligned}\zeta(s)&=\left(1+\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{8^s}+\dots\right)\cdot\\&\qquad\left(1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{27^s}+\dots\right)\cdot\\&\qquad\quad\left(1+\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{125^s}+\dots\right)\cdot\dots\\&=\prod_{p\text{ a prime}}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\dots\right)\\&=\prod_{p\text{ a prime}}\frac{1}{1-\frac{1}{p^s}}\end{aligned}$$

23. Suppose f is a multiplicative function and suppose g is defined by

$$\zeta(s)\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}$$

Show that

$$g(n)=\sum_{d|n}f(d)$$

24. Find a function u so that

$$\zeta(s)\sum_{n=1}^{\infty}\frac{u(n)}{n^s}=1$$

25. Find a function d so that

$$\sum_{n=1}^{\infty}\frac{d(n)}{n^s}=(\zeta(s))^2$$

26. Find a function g so that

$$\sum_{n=1}^{\infty}\frac{g(n)}{n^s}=\zeta(s)\cdot\zeta(s-1)$$

27. Find a function g so that

$$\zeta(s)\sum_{n=1}^{\infty}\frac{g(n)}{n^s}=\zeta(s-1)$$

28. Show that

$$\sum_{n=1}^{\infty}\frac{2^{\nu(n)}}{n^s}=\frac{\zeta(s)^2}{\zeta(2s)}$$

where $\nu(n)$ is number of distinct prime divisors of n.

29. Remember the function χ?

$$\chi(n)=\begin{cases}1 & \text{if } n\equiv 1 \pmod 4\\ -1 & \text{if } n\equiv 3 \pmod 4\\ 0 & \text{if } n \text{ is even}\end{cases}$$

Find a formula (or some other precise description) for the function b defined by

$$\zeta(s)\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\sum_{n=1}^{\infty}\frac{b(n)}{n^s}$$

30. If each sum converges, find its value (in terms of known functions). If it doesn't converge, prove it.

a.
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n}$$

b.
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^2}$$

c.
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^3}$$

d.
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^4}$$

e.
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^5}$$

f.
$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^6}$$

31. Use a high-powered CAS to investigate the values of ζ at integers,

$\longleftarrow$
Yes, ζ is, for us, a formal series, but it converges for some values of s.

And There's More . . .

Let $s_k(n)$ be the number of ways you can write n as a sum of k squares. Let $t_k(n)$ be the number of ways you can write n as a sum of k triangular numbers. Let $g(n)$ be the number of ways you can write n as $x^2 - xy + y^2$. And let $d_{k,m}(n)$ be the number of positive divisors of n that are congruent to k mod m.

Here are some things to investigate:

32.
$$s_2(n) = 4\left(d_{1,4}(n) - d_{3,4}(n)\right) = 4\sum_{d|n} \chi(d)$$

33.
$$s_2(n) = 4 \sum_{d|n, d \text{ odd}} (-1)^{\frac{d-1}{2}}$$

34.

$$s_4(n) = 8 \sum_{d\mid n, 4\nmid d} d$$

35.

$$t_2(n) = d_{1,4}(4n + 1n) - d_{3,4}(4n + 1)$$

36.

$$g(n) = 6\left(d_{1,3}(n) - d_{2,3}(n)\right)$$