Never mind that Journey, today we take it back to the old school...

13 Parents Just Don't Understand

Important Stuff

1. Use an Nspire to expand this if you haven't done it yet:

Expand the expression, not THIS.

$$(1 + 2x^1 + 2x^4 + 2x^9 + 2x^{16} + \cdots)^4$$

The coefficient of the x^n term gives the number of ways n can be written as the sum of four squares. We called this function s_4.

But its nickname was Mike.

(l) Determine $s_4(4)$.

(a) Find all the ways to write 4 as the sum of four squares (you don't need to write them all out). Order and signs matter, so $(-1)^2 + (-1)^2 + 1^2 + 1^2$ is different from $1^2 + (-1)^2 + 1^2 + (-1)^2$.

(na) Is s_4 multiplicative? Explain.

PROBLEM

Let $S_4(n) = \frac{s_4(n)}{8}$ and let R_4 be the parent of S_4. Fill in this table with the values of $R_4(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R_4(n)$	1			0	5										
$S_4(n)$	1	3	4	3	6	12	8	3	13	18	12	12	14	24	24

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$R_4(n)$															
$S_4(n)$	3	18	39	20	18	32	36	24	12	31	42	40	24	30	72

2. Compute the following product. Keep going until you notice something amazing!!

It takes Problem 2 to make a thing go right.

$$\left(\frac{1}{1^s} + \frac{2}{2^s} + \frac{3}{3^s} + \frac{0}{4^s} + \frac{5}{5^s} + \frac{6}{6^s} + \frac{7}{7^s} + \frac{0}{8^s} + \frac{9}{9^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) = \text{hmmmm}$$

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3. Look back at the last few days' expansion problems. Describe what happens when you multiply through by

$$\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)$$

What form must the other expression have for this to work?

4. Check out this table of values of the Möbius μ function.

Ch-ch-ch-check it out! Wha-what's it all about? Wor-wor-work it out! Let's turn this... never mind. This is the function we called "moo" on Monday.

n	factorization of n	$\mu(n)$	n	factorization of n	$\mu(n)$
1	1	1	16	2^4	0
2	2	-1	18	$2\cdot 3^2$	0
3	3	-1	20	$2^2\cdot 5$	0
4	2^2	0	21	$3\cdot 7$	1
5	5	-1	24	$2^3\cdot 3$	0
6	$2\cdot 3$	1	25	5^2	0
7	7	-1	30	$2\cdot 3\cdot 5$	-1
8	2^3	0	35	$5\cdot 7$	1
9	3^2	0	36	$2^2\cdot 3^2$	0
10	$2\cdot 5$	1	60	$2^2\cdot 3\cdot 5$	0
11	11	-1	77	$7\cdot 11$	1
12	$2^2\cdot 3$	0	99	$3^2\cdot 11$	0
14	$2\cdot 7$	1	210	$2\cdot 3\cdot 5\cdot 7$	1
15	$3\cdot 5$	1	2310	$2\cdot 3\cdot 5\cdot 7\cdot 11$	-1

Use the table to write a rule to calculate $\mu(n)$ for any n. Use your rule to calculate $\mu(120)$, $\mu(5005)$ and $\mu(30030)$.

Your rule for μ can be a sentence or two, it doesn't have to contain complicated symbols.

5. Look at the problem in the box from Day 11, then figure out the sequence of missing numerators in this equation below. Try to do it without performing any algebra.

That right side is better known as "1". What again did you say happens when you multiply through by $\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)$?

$$\left(\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\frac{?}{5^s}+\frac{?}{6^s}+\cdots\right)\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\cdots\right)$$

$$=\frac{1}{1^s}+\frac{0}{2^s}+\frac{0}{3^s}+\frac{0}{4^s}+\cdots$$

6. Find the first six numerators in this insane product of infinite sums. What does this have to do with today's problem in the box?

It's insane, got no brain!

$$\left(\frac{1}{1^s}+\frac{3}{2^s}+\frac{4}{3^s}+\frac{3}{4^s}+\frac{6}{5^s}+\frac{12}{6^s}+\cdots\right)\left(\frac{1}{1^s}+\frac{-1}{2^s}+\frac{-1}{3^s}+\frac{0}{4^s}+\frac{-1}{5^s}+\frac{1}{6^s}+\cdots\right)$$

$$=\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\frac{?}{5^s}+\frac{?}{6^s}+\cdots$$

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7. Patty believes that

$$1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \cdots = \left(1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{4^3} + \cdots\right) \cdot \left(1 + \tfrac{1}{9} + \tfrac{1}{81} + \tfrac{1}{9^3} + \cdots\right) \cdot \left(1 + \tfrac{1}{25} + \tfrac{1}{25^2} + \tfrac{1}{25^3} + \cdots\right) \cdot \left(1 + \tfrac{1}{49} + \tfrac{1}{49^2} + \tfrac{1}{49^3} + \cdots\right) \cdots \cdot \left(1 + \tfrac{1}{p^2} + \tfrac{1}{p^4} + \tfrac{1}{p^6} + \cdots\right) \cdots$$

Is she right? Why or why not?

This is another Overlong Product of Powers. The question is, though: are you down with OPP?

8. Here we go with another scenario! At least this time the sums are finite.

$$M = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) \cdots \left(1 - \frac{1}{p}\right) \cdots$$

Overlong products of primes! Yeah, you know me!

Wouldn't you know it, these just keep going on forever. Some more questions!!

(m) Will you find a $\frac{1}{15}$ term in this product? Why or why not? If so, whatzits sign?

(e) Will you find a $\frac{1}{18}$ term in this product? Why or why not? If so, whatzits sign?

(g) What's the sign of $\frac{1}{17}$?

(h) What happens with $\frac{1}{20}$? $\frac{1}{30}$?

(a) What denominators do you get, and with what signs?

(n) What is the result of the expansion? Holy cow!

Neat Stuff

9. So now let's look at $\frac{1}{M}$:

$$\frac{1}{M} = \left(\frac{1}{1 - \frac{1}{2}}\right)\left(\frac{1}{1 - \frac{1}{3}}\right)\left(\frac{1}{1 - \frac{1}{5}}\right)\left(\frac{1}{1 - \frac{1}{7}}\right) \cdots$$

Take each term from the original product and push it into the denominator. Add salt and pepper as needed.

Hey... wait a minute... these are all in the form $\frac{1}{1-r}$, like a geometric series! Whoomp!

There it is!

(j) Unravel each geometric series into its terms. For example, $\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

(o) Now multiply out the new $\frac{1}{M}$, if you haven't already.

(e) How big is $\frac{1}{M}$? What does that say about the value of M?!

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10. Repeat the last two problems with this infinite product:

$$N = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\cdots\left(1 - \frac{1}{p^2}\right)\cdots$$

What does the expansion of N look like? Find the value of N as exactly as you can by exploring the geometric series buried inside $\frac{1}{N}$. There has to be a value, since N must be between 0 and 1!

This is pretty tough, but if you aren't sure what comes next, bust a move back to the steps you followed in Problems 8 and 9.

Between 0 and 1 factorial, eh? It's like that, and that's the way it is.

11. (a) If two positive integers are picked at random, what is the probability that they are *both* multiples of 2?

(b) What is the probability that at least one of the two numbers *isn't* a multiple of 2? Psst: use $1 - p$.

(c) What is the probability that at least one of the two numbers *isn't* a multiple of 3?

(d) What is the probability that the two numbers don't have a common factor of 5? (This is the same as the last question.)

(e) 7? 11? Slurpee?

(f) What is the probability that two positive integers picked at random won't have a common factor of 2, 3, or 5?

(g) Write an expression for the probability that two positive integers picked at random will share *no* common factors.

This is the worst problem phrasing ever constructed, but that's how we roll. The choice is yours. Hey, we can't help it that 7 and 11 are consecutive primes.

12. What infinite series in the style of Problem 3 can be multiplied onto something to obtain its grandchild?

13. What infinite series in the style of Problem 3 can be multiplied onto something to obtain its grandparent?

Seth wonders why there are grandchildren and grandparents, but no grandmasters...

14. Let p be any prime. Complete this table.

"gp" is not a reference to Gangster's Paradise.

n	ggp	gp	parent of m	$m(n) = 1$	child of m	gc	ggc
1							
p							
p^2							
p^3							
p^4							

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15. Let p be any prime. Complete this table.

n	ggp	gp	parent of `id`	$\mathtt{id}(n) = n$	child of `id`	gc	ggc
1							
p							
p^2							
p^3							
p^4							

16. Show that for any integer $n \geq 2$,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

What do those symbols mean anyway?

17. Investigate the behavior of this function.

If you got to this problem, I got to say it was a good day.

$$f(n) = \frac{\sum_{k=1}^{n} \phi(k)}{n^2}$$

Tough Stuff

18. What is $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} s_4(k)$? Prove it.

19. Read the 14 proofs at this website:
`http://www.secamlocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf`
Which one do you like best? We hope at least one of them amazes you!

We like the proofs, the proofs that go boom. We're Sara and Maura and we like the boom.

20. Suppose you have an unlimited supply of beads with k different colors. How many distinct necklaces with length n can you make? Try to find a way to solve this problem using Möbius inversion.

If R and B are two colors (raw umber and burnt umber), RBRRR and RRBRR are not distinct necklaces since they are related by a circular shift.

21. Find a nice rule for $s_3(n)$, the number of ways to write n as the sum of three squares.

22. For what m is s_m proportional to a multiplicative function?

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