Your mathematical journey is nearing its end...

12 *The Parent Trap*

Important Stuff

PROBLEM

Now let's reach waaay back to Day 1. Remember the σ function? We defined it so that $\sigma(n)$ is the sum of the divisors of n. Let `id` be the parent of σ and `ego` be the grandparent of σ. Fill in this table using yesterday's parent-child connection.

n	1	2	3	4	5	6	7	8	9	10	11	12
`ego`(n)	1					2						
`id`(n)	1											
$\sigma(n)$	1	3	4	7	6	12	8	15	13	18	12	28

Sigma, sure, no problem! What? Yesterday's discussion with those lights might be helpful.

1. Find the first ten numerators in this zany product of infinite sums. What does this have to do with today's first problem in the box?

$$\left(\frac{1}{1^s}+\frac{2}{2^s}+\frac{3}{3^s}+\frac{4}{4^s}+\cdots\right)\left(\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right)$$
$$=\frac{?}{1^s}+\frac{?}{2^s}+\frac{?}{3^s}+\frac{?}{4^s}+\cdots$$

2. Consider this even zanier infinite product of infinite sums:

$$\begin{aligned}A = &\left(1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots\right)\\ &\cdot\left(1+\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\cdots\right)\\ &\cdot\left(1+\tfrac{1}{5}+\tfrac{1}{25}+\tfrac{1}{5^3}+\cdots\right)\\ &\cdot\left(1+\tfrac{1}{7}+\tfrac{1}{49}+\tfrac{1}{7^3}+\cdots\right)\cdots\\ &\cdot\left(1+\tfrac{1}{p}+\tfrac{1}{p^2}+\tfrac{1}{p^3}+\cdots\right)\cdots\end{aligned}$$

The only thing better than a zany product is an even zanier product! Anyway. Back to work.

Wow, this goes on forever. Let's think about what this expands *to*, not actually do it. The terms of the expansion come from picking one term from each set of parentheses, then multiplying them together. The final expansion is the sum of *all* such possibilities.

(s) Find a way to get $\frac{1}{12}$ by taking a piece from each factor.

(t) Is this the only way to get $\frac{1}{12}$?

(a) How many ways are there to get $\frac{1}{45}$?

(c) How many ways are there to get $\frac{1}{17}$?

(e) Pick another fraction in the form $\frac{1}{n}$ and describe how to get it in the expansion.

(y) What is the result of the expansion? How big is this product?

The product is big! So big, in fact, that it goes all the way to…We've seen the result of this expansion sometime in Week 1.

3. Consider this ridiculously zany infinite product of infinite sums of wacko numbers:

$$\begin{aligned} B = &\left(1 + \tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{4^3} + \cdots\right) \\ &\cdot \left(1 + \tfrac{1}{9} + \tfrac{1}{81} + \tfrac{1}{9^3} + \cdots\right) \\ &\cdot \left(1 + \tfrac{1}{25} + \tfrac{1}{25^2} + \tfrac{1}{25^3} + \cdots\right) \\ &\cdot \left(1 + \tfrac{1}{49} + \tfrac{1}{49^2} + \tfrac{1}{49^3} + \cdots\right) \cdots \\ &\cdot \left(1 + \tfrac{1}{p^2} + \tfrac{1}{p^4} + \tfrac{1}{p^6} + \cdots\right) \cdots \end{aligned}$$

Dang, this goes on forever *too*. Let's ask some incredibly similar questions.

(r) Find a way to get $\frac{1}{144}$ by taking a piece from each factor.

(i) Is this the only way to get $\frac{1}{144}$?

(c) How many ways are there to get $\frac{1}{45^2}$?

(h) How many ways are there to get $\frac{1}{17^2}$?

(a) How many ways are there to get $\frac{1}{20}$? Why?

(r) Pick another fraction in the form $\frac{1}{n^2}$ and describe how to get it in the expansion.

(d) What is the result of the expansion? How big is this infinite product of infinite sums? Infinite, right? Riii-iight?

It's only in this last part that this problem and the last go in separate ways. We've seen the result of this expansion sometime in Week 2.

PROBLEM

Yesterday, we defined s_2, a function counting how many ways you can write numbers as the sum of two squares. We noted that s_2 itself isn't multiplicative since $s_2(1) = 4$, but that $S_2(n) = s_2(n)/4$ seems to be multiplicative. Let R_2 be the parent of S_2. Fill in this table with the values of $R_2(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$R_2(n)$	1		-1						1						
$S_2(n)$	1	1	0	1	2	0	0	1	1	2	0	0	2	0	0

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$R_2(n)$															
$S_2(n)$	1	2	1	0	2	0	0	0	0	3	2	0	0	2	0

Turns out you can find a parent regardless of whether or not the original function is multiplicative. As we'll see though, if the parent is multiplicative, the child is just the same way. Isn't the R_2 function amazing??

4. (a) Write a simple rule that could be used to calculate $R_2(n)$ for any n.

(b) Calculate this:

$$R_2(1) + R_2(2) + R_2(5) + R_2(10) + R_2(13) + R_2(26) + R_2(65) + R_2(130)$$

(c) What is $S_2(130)$? Calculate any way you want it.

Is that the way you need it? Sorry, this joke was written far too late at night.

5. Using what you know about the R_2 function, find and justify a rule for $S_2(p)$ for prime p.

6. Find and justify a rule for $S_2(n)$ for any n. Your work in Problem 4 may help.

7. Suppose that f is a non-zero multiplicative function and g is its child. Let $f(3) = a$ and $f(7) = b$.

(a) What is the only possible value for $f(1)$?

(b) Calculate $f(21)$ in terms of a and b.

(c) Write $g(3)$ in terms of a. Remember, g is the child of f.

(d) Write $g(7)$ in terms of b.

(e) Write $g(21)$ in terms of a and b.

(f) Is it true that $g(21) = g(3)g(7)$? For what kind of numbers could this argument be used?

See Problem 3 on Day 11, people.

The parent-child relationship means that $g(3) = f(1) + f(3)$. This will help you find the only solutions to each case.

Neat Stuff

8. Let p be any prime. Complete this table.

n	$\sigma(n)$	$\mu(n)$	$\phi(n)$	$\tau(n)$
1				
p				
p^2				
p^3				
p^4				

OK, that $moo(n)$ function from yesterday? It's really called the Möbius μ function. Please accept this small change with open arms.

9. A *lattice point* is a point with integer coordinates. How many lattice points are on the graph of each of these?
(a) $x^2 + y^2 = 25$
(b) $x^2 + y^2 = 65$
(c) $x^2 + y^2 = 1105$

If you held these graphs above your head, each one would look like a wheel in the sky.

10. Figure out the sequence of missing numerators. Can you do it without performing any algebra?

$$\left(\frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)$$

$$= \frac{1}{1^s}+\frac{1}{2^s}+\frac{0}{3^s}+\frac{1}{4^s}+\frac{2}{5^s}+\frac{0}{6^s}+\frac{0}{7^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{2}{10^s}+\frac{0}{11^s}+\frac{0}{12^s}+\cdots$$

11. Again, figure out the sequence of missing numerators. Holy cow, there's only possible answer here.

$$\left(\frac{?}{1^s} + \frac{?}{2^s} + \frac{?}{3^s} + \frac{?}{4^s} + \cdots\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) = 1$$

Work faithfully on this problem and a reward awaits!

12. Define $s_4(n)$ to be the number of ways to write n as the sum of four squares, where the order and signs of numbers matters. For example, $s_4(1) = 8$ because

$$\begin{aligned}
1 &= (\pm 1)^2 + 0^2 + 0^2 + 0^2 \\
1 &= 0^2 + (\pm 1)^2 + 0^2 + 0^2 \\
1 &= 0^2 + 0^2 + (\pm 1)^2 + 0^2 \\
1 &= 0^2 + 0^2 + 0^2 + (\pm 1)^2
\end{aligned}$$

If you already did this problem on Day 11, look into the future and skip this.

Your mathematical journey is nearing its end...

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_4(n)$	1	8	24	32	24	48	96	64	24	104	144	96	96

n	13	14	15	16	17	18	19	20	21	22	23	24	25
$s_4(n)$	112	192	192	24	144	312	160	144	256	288	192	96	248

(a) Determine whether or not s_4 is multiplicative.
(b) Define a function S_4 based on s_4 that you think is multiplicative, and test a few examples.

13. Make a conjecture about the value of $S_4(p)$ for prime p.

14. Here's an interesting sequence o' sequences.

Step	Sequence
1	1 1
2	1 2 1
3	1 3 2 3 1
4	1 4 3 2 3 4 1
5	1 5 4 3 5 2 5 3 4 5 1

At step n, look through the last sequence for two consecutive numbers that add to n, and whenever that happens, insert n. Investigate this and look for any interesting connections.

Bowen is loving this problem. Try keeping track of the number of insertions. Say, this might even connect with the circles that were touching each other, when we kept squeezing more into the diagram.

15. As we did with s_2 and s_4, define $s_3(n)$ as how many ways you can write n as the sum of *three* squares (with positions and signs of the three numbers being significant).
(a) Use a power series to help you generate data for s_3 quickly. Or, construct a three-dimensional version of Day 11's handout. Your choice!
(b) Is s_3 multiplicative or can it made multiplicative like we did S_2 and S_4?
(c) Determine r_3, the parent of s_3. See anything cool?

I wonder who's crying now after working on this problem too hard.

16. Use the style of Problem 7 to prove more generally that if g is the child of f and f is multiplicative, then so is g.

Tough Stuff

17. Prove that if g is the child of f and f is *not* multiplicative, then neither is g. Hint: find the smallest n that violates the multiplicativity of f and...

Never mind that Journey, today we take it back to the old school...

The Parent Trap

18. Categorize all positive integers n that *cannot* be written as the sum of three squares.

19. Prove that any positive integer n can be written as the sum of four squares.

20. Find the exact value of each summation, or show that the sum diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n^2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\tau(n)}{n^2}$

21. Define $s_m(n)$ to be the number of ways n can be written as the sum of m squares, where the order and signs of the numbers matters. For which positive integers m is s_m proportional to a multiplicative function?

Don't stop believing that there will be more math problems and bad jokes and references each day until Friday.