Have an awe-sum weekend!

5 *Welcome to Sum-it County*

Important Stuff

1. The ϕ function gives the number of values from 1 to n that are relatively prime to n; that is, those number that share no common factors greater than 1 with n. For example, $\phi(6) = 2$ since 1 and 5 don't share common factors with 6 (besides 1), while 2, 3, 4 do.

(a) Determine $\phi(3)$, $\phi(5)$, and $\phi(15)$.

(b) Determine $\phi(2)$, $\phi(7)$, and $\phi(14)$.

(c) If p is a prime number, what is a formula for $\phi(p)$?

(d) Determine a formula for $\phi(pq)$ where p and q are distinct primes.

Here's a handy-dandy table for $\phi(n)$.

Oh it will be useful... soon! All too soon.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(n)$	1			2		2		4	6	4	10	4
n	13	14	15	16	17	18	19	20	21	22	23	24
$\phi(n)$	12			8	16	6	18	8	12	10	22	8

2. Let $f(x) = \dfrac{1}{x}$. The *child* g of f is found by calculating

$$\begin{aligned} g(n) &= f(\text{all divisors of } n) \text{ added together} \\ g(15) &= f(1) + f(3) + f(5) + f(15) \\ g(20) &= f(1) + f(2) + f(4) + f(5) + f(10) + f(20) \\ g(1) &= f(1) \end{aligned}$$

(a) Determine $g(3)$, $g(5)$, and $g(15)$.

(b) Determine $g(2)$, $g(7)$, and $g(14)$.

(c) If p is a prime number, what is a formula for $g(p)$?

Thanks for all your hard work and insight this week! We hope you've had a good time and learned a few things... more to come.

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PROBLEM

Let r be the child of ϕ, which we defined in Problem 1. Remember, if r is the child of ϕ it means that $r(3) = \phi(1) + \phi(3)$ and $r(10) = \phi(1) + \phi(2) + \phi(5) + \phi(10)$, among others.

Too soon?

Complete this table for $r(n)$ from 1 to 24, and determine a rule for $r(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$r(n)$												

n	13	14	15	16	17	18	19	20	21	22	23	24
$r(n)$												

Cool, huh?

3. Determine the grandchild of the ϕ function by finding the child of r.

4. Find all solutions for each of the following.
(a) $x^2 = x$ in mod 3
(b) $x^2 = x$ in mod 5
(c) $x^2 = x$ in mod 7
(d) $x^2 = x$ in mod 15
(e) $x^2 = x$ in mod 21
(f) $x^2 = x$ in mod 35
(g) $x^2 = x$ in mod 105

Be careful on the fourth!

5. List five things you learned this week.

Mathematical things, not facts like "Oscar's favorite color is puce."

Neat Stuff

6. What is the child of $f(n) = 1$, a constant function?

7. Which of the following functions are multiplicative?
(a) $f(n) = n^2$
(b) $g(n) = 2n$
(c) $m(n) = n \bmod 12$
(d) $a(n) = \frac{\sigma(n)}{n}$
(e) $y(n) = \gcd(n, 12)$

8. If f is a multiplicative function, what are the possible values of $f(1)$?

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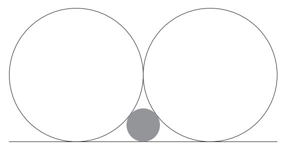
9. Is the sum of two multiplicative functions also multiplicative? What about the product?

While no proof is expected here, you might think about how you would prove some of these. How are multiplicative functions "built"?

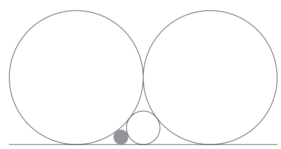
10. Is the child of a multiplicative function also multiplicative?

11. Two circles with diameter 1 are tangent to a line, as well as to each other. What is the diameter of the gray circle that is tangent to both circles as well as the line?

What is this really a picture of? Someone smoking while looking through binoculars? An orange stuck in the cantaloupe aisle?



12. Start with the diagram from the previous problem, and add one more circle in the hole in the lower left; it is also tangent to the line and two other circles. What is the diameter of this circle (marked in gray below)?



13. The function $k_5(n)$ counts the number of solutions to the equation $xy = 5 \bmod n$. Here's a table of some values of $k_5(n)$.

Wouldn't you really rather do this by making a giant chart again? Really? You sure?

n	1	2	3	4	5	6	7	8	9	10	11	12
$k_5(n)$	1	1	2	2	9	2	6	4	6	9	10	4
n	13	14	15	16	17	18	19	20	21	22	23	24
$k_5(n)$	12	6	18	8	16	6	18	18	12	10	22	8

Find a rule for the *child* of k_5.

14. Prove that if a and n are relatively prime, then there is a solution to the equation $ab = k$ in mod n for every number k from 0 to $n - 1$.

Eat Pizza, Do Math: One method uses a pigeonhole argument. What would happen if one of the equations couldn't be solved? Use that to find a contradiction.

15. Find all solutions to each of the following equations.

(a) $x^2 = 1$ in mod 3
(k) $x^4 = 1$ in mod 5
(e) $x^5 = 1$ in mod 7
(m) $x^{10} = 1$ in mod 11
(i) $x^5 = 1$ in mod 6

16. If $x^2 = 1$ in mod p for prime p, what are the possible values of x?

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17. In mod 11, the perfect squares are 1, 4, 9, 5, and 3. Find the perfect squares in each of these mods.
(a) mod 7
(b) mod 13
(c) mod 17
(d) mod 19

And the perfect strangers are Larry and Balki.

18. Find all solutions to each of the following equations.
(a) $x^3 = 1$ in mod 7
(b) $x^3 = -1$ in mod 7
(c) $x^6 = 1$ in mod 13
(d) $x^8 = 1$ in mod 17
(e) $x^9 = 1$ in mod 19

Ooh, a scavenger hunt question.

19. Without technology, find all the solutions to the equation $x^2 - 4x + 3 = 0$ in mod 165.

Aretha says you better think about what you're trying to do to this before brute-forcing it.

20. Let $c(n)$ be the number of solutions to the equation $x^2 = 1$ in mod n. Prove that c is multiplicative.

Tough Stuff

21. Determine the grandparent of $f(n) = 1$.

22. Problem 13 introduced a function called $k_5(n)$ and asked you to find a rule for its child. Find a more general rule for the child of $k_a(n)$, the number of solutions to the equation $xy = a$ in mod n. This rule should work for $a = 0$ and $a = 1$, among others.

23. Let $S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ be the sum of the reciprocals of all odd numbers. Determine, with proof, whether this sum converges or diverges.

24. Let $S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ be the sum of the reciprocals of all prime numbers. Determine, with proof, whether this sum converges or diverges.

25. Let $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{12} + \frac{1}{14} + \cdots$ be the sum of the reciprocals of all numbers *that don't have a 3 in them*. Determine, with proof, whether this sum converges or diverges.

26. Find an odd number such that $\sigma(n) = 2n$, or prove that no such number exists.