$5^2 - \omega^3$ hours to go, I wanna be sedated

14 *pi-o-neer day*

Important Stuff

PROBLEM

Find Part 2 of the "problem out of the box" from Day 11. First, check your numbers with all of the people around your table. Then, add one more row at the bottom of your chart like this.

97	98	99	100	101	102	103	104	105	106	107	108

Count the number of times the numbers 97 through 108 appear on the square grid of circles from Part 1. Finally, find the average of the numbers you wrote in all 108 boxes. Report this average to three decimal places.

If a number doesn't show up on the square grid of circles, make sure you include that 0 in your average.

1. (c) Draw a circle with radius 5 centered at the origin in the xy-plane. Exactly how many coordinates (x, y) with integers x and y are inside or on the circle?

(h) Count the number of times that the numbers from 1 to 25 appear on your square grid of circles (Part 1 from "problem out of the box" from Day 11).

(r) Draw a circle with radius $5\sqrt{2}$ centered at the origin in the xy-plane. Exactly how many coordinates (x, y) with integers x and y are inside or on the circle?

What's the equation for that circle?

(i) Count the number of times that the numbers from 1 to 50 appear on your square grid of circles.

(s) Estimate the number of times that the numbers from 1 to N will appear on an infinite square grid of circles filled in with the numbers $x^2 + y^2$.

2. Imagine extending today's problem in the box to compute the *average* of the number of times that the numbers from 1 to 1000 appear on an infinite square grid of circles filled in with $x^2 + y^2$. What do you expect the answer to be?

Just imagine—don't do this by hand. You'll be here a long time.

Neat Stuff

3. Pick a positive integer at random. What is the expected number of ways that the integer can be expressed as the sum of two squares?

4. Find Part 2 of the "problem out of the box" from Day 12. First, check your numbers with all of the people around your table. Then, find the average number of times that the numbers 1 to 96 appears on the isometric grid of circles. Report this average to three decimal places. (If a number doesn't show up on the grid of circles, make sure you include that as 0 in your average.)

What number is that?! You won't have enough digits of accuracy for Plouffe's Inverter to be of any help.

5. If you haven't finished problems 8 and 13 from Day 13 yet, do them. They're fun!

6. Determine if each of these Gaussian integers is prime. If it is prime, explain how you know it is prime. If it is composite, give at least one non-boring factorization of it.

(p) $1 + i$
(a) $10 + 10i$
(t) $7i$
(r) $-i$
(i) $2 - 3i$
(c) -13
(k) $-10 + 13i$

7. (a) Plot the graph of $x^2 + y^2 - xy = 1$ in the xy-plane.
(b) What is the area of an ellipse with semi-major axis length a and semi-minor axis length b?

8. (a) Plot the graph of $x^2 + y^2 - xy = 1$ in the isometric dot plane.
(b) What is the area of the parallelogram created by the points $[\rightarrow 0, \nwarrow 0]$, $[\rightarrow 0, \nwarrow 1]$, $[\rightarrow 1, \nwarrow 1]$, $[\rightarrow 1, \nwarrow 0]$ in the isometric dot plane?

9. Let $t_w(n)$ be the number of times that the positive integer n can be written as x^2+y^2-xy, where x and y are integers. The function $t_w(n)$ corresponds to the numbers you wrote in Part 2 of the "problem out of the box" for Day 12. Find

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} t_w(n).$$

Either of the previous two problems may help you solve this problem.

10. Determine if each of these Eisenstein integers is prime. If it is prime, explain how you know it is prime. If it is composite, give at least one non-boring factorization of it.
(l) $2 + w$
(o) $5w$
(u) $1 + 5w$
(i) $-4 + 7w$
(s) $-4 - 7w$

11. Let $t(n)$ be the number of times that the positive integer n can be written as $x^2 + y^2$, where x and y are integers. Explore how

$$\frac{1}{N}\sum_{n=1}^{N} t(n)$$

behaves as a function of N. What value(s) of N under 200 give the closest approximation to the limiting value as $N \to \infty$?

This problem is meant for people who like geeking out on their calculators or computers. Finally, we're giving you a chance to pull out your Nspire, Excel spreadsheets, Maple, Mathematica, or whatever else, and have fun(?).

12. Suppose $N(a+b\sqrt{2}) = k$ for some integers a, b. Show that $(a^2 + 2ab, a^2 + 2ab - k, a^2 + 2ab + 2b^2)$ is a Pythagorean triple.

13. Use problem 12 and the method from the problem in the box for Day 10 to find some Pythagorean triples that are really, really close to being isosceles right triangles.

Are you really, really close, or just really close? We want really, really close!

14. Find the smallest positive integer that has 4 factors that are congruent to 1 mod 4, and 1 factor that is congruent to 3 mod 4. Or, prove that no such number exists.

15. Find the lengths of the sides of the triangle formed by the thre epoints $(-18, 49)$, $(15, -7)$, and $(30, -15)$. Find a way to generate other triangles with vertices on lattice points in the xy-plane, side lengths that are integers, and sides that are neither horizontal nor vertical.

16. (t) How many *distinct* peg-to-peg lengths can be found on a $n \times n$ piece of square dot paper?
(o) How many *distinct* peg-to-peg lengths can be found on a $n \times n$ piece of isometric dot paper?
(d) How many distinct, *integer* peg-to-peg lengths can be found on a $n \times n$ piece of square dot paper?
(d) How many distinct, *integer* peg-to-peg lengths can be found on a $n \times n$ piece of isometric dot paper?

Tough Stuff

17. Find some Pythagorean triples that are really, really close to being 30°-60°-90° triangles.

18. Find some Eisenstein triples that are really, really close to being 30°-60°-90° triangles.

19. Look for patterns in the tables from problems 16 and 17 on Day 13. Then, prove them.

If you think this problem is hard, try milking a wild cow.

20. Pick a positive integer at random. What is the probability that it can be written as the sum of two squares?

21. You know Pick's Theorem, right? If not, figure out what it is. Your job in this problem is to generalize Pick's Theorem so that works in three dimensions. Is there a version of the isometric dot paper that works in three dimensions?