

# 10

## Wall•i



### Important Stuff

1. (i) Use  $4 + i$  to generate a Pythagorean triple.
- (ii) Use  $4 + w$  to generate an Eisenstein triple.
- (iii) Get out yet another sheet of square/isometric dot paper. On the square dot paper, pick a point as your origin, and plot  $4 + i$  and  $(4 + i)^2$ . Draw triangles that correspond to both numbers and label their side lengths. Turn the paper over and repeat for  $4 + w$  and  $(4 + w)^2$ .

OK, these triples are supposed to be called *Eisenstein triples*. Bowen claims the "se" is silent. Eisenstein triples are positive integers with the property that they can be the side lengths of a triangle with a  $60^\circ$  angle. Eisenstein triples relate to triangles with a  $60^\circ$  angle in the same way that Pythagorean triples relate to triangles with a  $90^\circ$  angle. "Franz" Ferdinand Gotthold Max Eisenstein (1823-1852) was a student of Carl Friederich Gauss. We are also trying to go for the record for the longest side note here. Woo hoo!

$N$  is your friend. It takes a number and multiplies it by its conjugate.

In  $\sqrt{2}$ -land, the conjugate of  $a + b\sqrt{2}$  is  $a - b\sqrt{2}$ , so  
 $N(5+2\sqrt{2}) = (5+2\sqrt{2})(5-2\sqrt{2}) = 17$ .

### PROBLEM

Something interesting happens when we raise  $1 + \sqrt{2}$  to different powers.

$$\begin{aligned} (1 + \sqrt{2})^1 &= 1 + \sqrt{2} & \text{and} & & 1/1 &= 1 \\ (1 + \sqrt{2})^2 &= 3 + 2\sqrt{2} & \text{and} & & 3/2 &= 1.5 \\ (1 + \sqrt{2})^3 &= 7 + 5\sqrt{2} & \text{and} & & 7/5 &= 1.4 \\ (1 + \sqrt{2})^4 &= ? \end{aligned}$$

If you continue this pattern, what do you notice about the fractions? What's going on here?  $N$ -values anyone?

2. Here are some fun facts:  $N(3+\sqrt{2})=7$  and  $N(3+2\sqrt{2})=1$ .
  - (a) Explain why  $N((3 + \sqrt{2})^2) = 49$ .
  - (b) What's  $N((3 + \sqrt{2})^3)$ ?
  - (c) What's  $N((3 + 2\sqrt{2})^n)$  for any positive integer  $n$ ?
3. Repeat the problem in the box, but start with these numbers instead of  $1 + \sqrt{2}$ . What happens to the fractions?
  - (a)  $3 + 2\sqrt{2}$
  - (b)  $2 + \sqrt{3}$
  - (c)  $2 + \sqrt{5}$
  - (d)  $3 + \sqrt{10}$

Fun is a relative thing.

Get back to work!

In this version of the movie, the main character will be played by a fish.

4. Find integers  $x$  and  $y$  with  $x > 1$  so that  $N(x + y\sqrt{7}) = 1$ . Use these numbers to make a fraction that gives a really good approximation to  $\sqrt{7}$ . Come on! You can make a better approximation than that! Ask Peg how to use the magic box.
5. If you are given a triple of numbers, how can you test whether it is a Pythagorean triple? An Eisenstein triple?
6. (a) What does the value of  $N(5 + 2i)$  have to do with the distance from the origin to  $5 + 2i$  on the complex plane?  
(b) What does the value of  $N(5 + 2w)$  have to do with the distance from the origin to  $5 + 2w$  on the complex plane?
7. Why do we *square* numbers of the form  $a + bi$  or  $a + bw$  to obtain Pythagorean triples or Eisenstein triples? Why *squaring* as opposed to taking a cube root, or reciprocal, or *(insert obscure math operation here)*?

### Old Stuff

8. Which positive integers  $n$  have we determined cannot be written as  $x^2 + y^2$  with  $x$  and  $y$  being integers? Look at the problem in the box for Day 4. There may be other numbers that can't be written as  $x^2 + y^2$  or  $x^2 + y^2 - xy$  that we haven't identified yet.
9. Which positive integers  $n$  have we determined cannot be written as  $x^2 + y^2 - xy$  with  $x$  and  $y$  being integers?
10. What is  $i^4$ ? What is  $w^3$ ?
11. Find integers  $a$  and  $b$  so that  $N(a + bw) = 7$ . Find integers  $c$  and  $d$  so that  $N(c + dw) = 13$ . Calculate  $(a + bw)(c + dw)$ , then calculate  $N((a + bw)(c + dw))$ .
12. Find an Eisenstein triple  $(a, b, c)$  with  $c = 91$ . Wait a minute! This number didn't go up!
13. Calculate the  $N$ -value of these numbers.  
(a)  $3 + 2\sqrt{2}$   
(b)  $3 + 2i$   
(c)  $3 + 2w$
14. Rewrite these without using fractions. OK, OK. We know  $w$  has a fraction in it. That one doesn't count in your answer.
- (a)  $\frac{11 + 7\sqrt{2}}{3 + 2\sqrt{2}}$                       (c)  $\frac{17 + 9w}{3 + 2w}$
- (b)  $\frac{17 + 7i}{3 + 2i}$                               (d)  $\frac{290 + 190w}{29 + 19w}$

### Neat But Potentially Useless Stuff

15. For each of the following values of  $n$ , find integers  $x$  and  $y$  with  $x > 1$  so that  $N(x + y\sqrt{n}) = 1$ . Use these numbers to make a fraction that gives a really good approximation to  $\sqrt{n}$ .

- (a) 6
- (b) 11
- (c) 17
- (d) 529
- (e) 85

Here are a bunch of Eisenstein triples. In other words, these numbers can be the lengths of a triangle with a  $60^\circ$  angle. The third number is the length opposite the  $60^\circ$  angle.

(5, 8, 7) (7, 15, 13)  
 (11, 35, 31) (16, 21, 19)  
 (33, 40, 37) (32, 77, 67)  
 (117, 40, 103) (13, 48, 43)  
 (57, 112, 97) (39, 55, 49)  
 (79, 79, 79)  
 (799, 799, 799)  
 (7999, 7999, 7999)

Those last three are boring.

16. In an Eisenstein triple, why is the length of the side opposite the  $60^\circ$  angle usually in between the other two lengths? What kind of triangles are exceptions to this pattern? There are a bunch of Eisenstein triples over there.  $\rightarrow$

17. On Day 5, we noticed this algebraic identity having to do with Pythagorean triples and right triangles.

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

Is there an analogous identity for Eisenstein triples?

Refer to the whiteboard notes posted on our 2008 SSTP site. Convince yourself this identity is true (algebraically *and* geometrically) if you haven't already done so.

18. Pick some integers  $a$  and  $b$  and square  $a + bi$  and  $a + bw$  to generate some Pythagorean or Eisenstein triples. You may find that sometimes the numbers in the triple have a common factor. How can you choose  $a$  and  $b$  so that you get a primitive Pythagorean triple? What about a primitive Eisenstein triple?

In this context, a *primitive* triple of numbers is one that has no common factors.

19. Does squaring  $a+bw$  give you all possible Eisenstein triples?
20. Figure out a way to generate triangles with integer side lengths that have a  $120^\circ$  angle.
21. Use what you know about the  $N$  function to factor  $74+7i$ .
22. Does anything unusual or special happen when you cube numbers of the form  $a + bi$  and  $a + bw$ ?
23. Plot the graph of  $x^2 + y^2 - xy = 49$  on isometric dot paper.
24. Andrew says that he has an identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (bc + ad)^2.$$

Is this identity true? How is this identity related to complex numbers?

### Tough Stuff

25. Find a distance so that more than 24 points on a isometric dot plane are that distance away from the origin.
26. Find a distance so that more than 24 points on a square dot plane are that distance away from the origin.
27. Find a distance so that more than 24 points on *both* square and isometric dot planes are that distance away from the origin.
28. Derive the *w*-world version of the identity in problem 24.
29. Find two positive integers  $a$  and  $b$  such that when you make a right triangle with these two numbers as legs the hypotenuse is an integer, *and* when you make a triangle with a  $60^\circ$  angle with these two numbers as side lengths the side opposite the  $60^\circ$  angle is an integer. Or, prove that no such  $a$  and  $b$  exist.
30. The current record for calculating  $\pi$  is into the trillions of digits. One completely wacky formula for computing  $\pi$  comes from this calculation:

$$\frac{(Q + i)^{12} \cdot (57 + i)^{32} \cdot (110443 + i)^{12}}{(239 + i)^5}$$

Oh wait, we forgot  $Q$ . Find the integer  $Q$  so that this entire number makes an  $45^\circ$  with the positive real axis when plotted in the complex plane. This formula was discovered by a high school teacher in 1982. Perhaps you could find some more?

Just understanding this problem statement is already an accomplishment... If you can rewrite this problem to make it easier to understand, please write it on the back of a \$100 and give it to us.

Check out problem 19 on Day 5.