It's $MO + N(d + ai)$! Yay!

6 *i's, i's, baby*

Important Stuff

PROBLEM

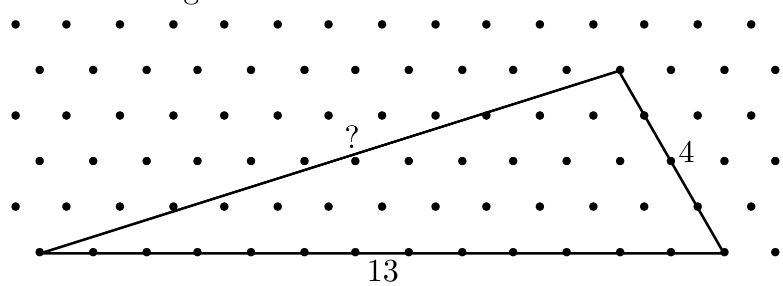
Find the sixteen complex numbers $x + yi$ with integers x, y that have

$$N(x + yi) = 65$$

and plot them on a complex plane. What do you notice?

In case you played too hard this weekend and can't remember, we defined N to be a function takes a number and multiplies it by its conjugate. $N(3+2i) = (3+2i)(3-2i) = 13$

1. This triangle was drawn on isometric dot paper. What is the exact length of the third side?



This question should use trigoNOmetry. As in, don't use that. That includes the Flaw of Cosines.

2. Find a and b so that $N(a + bi) = 17$. Find c and d so that $N(c + di) = 5$. Calculate $(a + bi)(c + di)$, then calculate $N((a + bi)(c + di))$.

Oh yeah, a, b, c and d are supposed to be integers.

3. Use the results of problem 2 to find a Pythagorean triple with hypotenuse 17. Then find a triple with hypotenuse 5. And finally, 85.

4. Which of these points are one unit away from the origin?

(a) $(1, 0)$
(b) $(1/2, 1/2)$
(c) $(3/5, 4/5)$
(d) $(-5/13, 12/13)$

5. Write a rule you could use to determine if a point (x, y) is one unit away from the origin. Sketch a graph of all points (x, y) that are one unit away from the origin.

6. Find all the numbers (real and complex) that solve the equation $z^4 = 1$. Did you find them all? Plot the solutions on the complex plane.

7. Calculate and plot these on a complex plane.

nv s ble man

(a) i^2
(b) i^3
(c) i^4
(d) i^5
(e) i^6
(f) i^7
(g) i^8
(h) i^{299}
(i) i^{300}
(j) i^{301}

8. Let $b = 4 + i$, $e = 2 + i$, $v = 4 + 6i$.

You are the apple π of my i.

(a) Plot and label b, e, and v in a complex plane.
(b) Multiply b, e, and v by i and plot those points in the same plane.
(c) Multiply b, e, and v by i *twice* and plot those points in the same plane.
(d) What's going on here?

9. (a) Show that $z^3 - 1 = (z - 1)(z^2 + z + 1)$.
(b) Find all three solutions to the equation $z^3 = 1$.
(c) Plot all of your solutions carefully on the complex plane. You may need to use a calculator to help you find their locations on the plane.

"Have fun storm$i$ng the Bast$i$lle!"

10. Let $w = \dfrac{-1 + i\sqrt{3}}{2}$. Calculate and plot these.

(j) w^2
(a) w^3
(n) w^4
(i) w^{299}
(e) w^{300}
(c) w^{301}
(e) $(w^2)^3$ (So, what equation does w^2 solve?)

Neat Stuff

11. Use the method of problem 2 to find a Pythagorean triple with hypotenuse 145. Then 221. And finally, 1105!

12. Find other triangles that can be drawn on isometric dot paper like the one in problem 1 that have a 60° angle, two sides with integer lengths, and a third side with the same length as the missing side in problem 1.

13. Find a piece of isometric dot paper. Pick a dot in the center and call it O. How many points can you find that are the same distance away from O as the distance you found in problem 1?

14. Shoba stands at the origin $(0, 0)$ and stares at $1+i$, $(1+i)^2$, $(1+i)^3$ and so on. Describe what happens to the powers of $1+i$ from her perspective: where do they go? how far away (N-value, anyone)?

15. Let $b = 4 + i$, $e = 2 + i$, $v = 4 + 6i$.
(a) Plot and label b, e, and v in a complex plane.
(b) Multiply b, e, and v by $1 - 3i$ and plot those points in the same plane.
(c) Find the area of the new triangle formed by these three points.

z-plane! z-plane!

16. Find the intersection(s) of the unit circle and these lines.
(a) $y = 2x - 1$
(b) $3x - 2y = 2$
(c) $y + 1 = 4x$

17. Get some graph paper and draw a really large unit circle. No, really, I mean a *really* large circle. Let $x = 3/5$ and $y = 4/5$ and plot (x, y). Use the transformation

$$(x, y) \mapsto (x^2 - y^2, 2xy)$$

and plot your answer. Then do the same thing with the point you just plotted. Repeat ad nauseum. What happens?

A gaggle of geese...
A covey of quail...
A murder of ravens...
A faction of fractions?

...besides you getting nauseous.

18. Write each prime as $n = x^2 + y^2 - xy$, where x and y are integers, or determine that it's impossible.
(l) 103
(a) 107
(u) 109
(r) 1009
(i) 4003
(e) 11111111111111111111111

Oh yes, there are 23 ones right there.

19. Does squaring $a + bi$ give you all the Pythagorean triples?

20. How many different peg-to-peg squares are there on an $n \times n$ piece of square dot paper?

Oh noes: starting with 3×3, there are bonus squares!

21. How many different peg-to-peg squares are there on an $n \times n$ piece of isometric dot paper?

It's $MO + N(d + ai)$! Yay!

i's, i's, baby

22. How many different peg-to-peg equilateral triangles are there on an $n \times n$ piece of isometric dot paper?

Bonus triangles!!!

Tough Stuff

23. Using only geometry and no algebra (no quadratic formula, no completing the square), find all complex numbers a and b that solve

$$\begin{aligned} a + b &= -1 \\ ab &= 1. \end{aligned}$$

Absolutely trigoNOmetry!

24. Prove that every positive integer not of the form $8n + 7$ or $4n$ is a sum of three squares having no common factor.

Legend(re) proved this in 1798.

25. Suppose n is a positive integer. Is there a right triangle with rational numbers as its side lengths with area n?

26. Is there a generalization of Pick's Theorem for isometric geoboards?