# 4 *i Am Legend(re)*

**Important Stuff**

**PROBLEM**

Fill in this table with $x^2 + y^2$ mod 3.

| $y$ \ $x$ | 0 | 1 | 2 |
|---|---|---|---|
| 2 | | | |
| 1 | | | |
| 0 | | | |

Fill in this table with $x^2 + y^2$ mod 4.

| $y$ \ $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | | | | |
| 2 | | | | |
| 1 | | | | |
| 0 | | | | |

Fill in this table with $x^2 + y^2 - xy$ mod 3.

| $y$ \ $x$ | 0 | 1 | 2 |
|---|---|---|---|
| 2 | | | |
| 1 | | | |
| 0 | | | |

Fill in this table with $x^2 + y^2 - xy$ mod 4.

| $y$ \ $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 3 | | | | |
| 2 | | | | |
| 1 | | | | |
| 0 | | | | |

"Mod 4" is just a fancy way of saying "remainder after dividing by 4," just like "fuschia" is a fancy way of saying "purple."

**1**. Prove using one of the charts above that none of Allen's four kids' ages can be written in the form $x^2 + y^2 - xy$.

Errr... $x$ and $y$ should be integers. When these problems were written, Allen's kids' ages were 2, 5, 8, and 11.

**2**. Figure out a way to rewrite these without using fractions.

(a) $\dfrac{8+i}{3+2i}$ (c) $\dfrac{43+6i}{7+4i}$

(b) $\dfrac{8+i}{2-i}$ (d) $\dfrac{130+20i}{13+2i}$

Ben found the names Ted and Dan while rearranging the letters in "rat the den." Darryl thinks everyone who lives in 936 is addicted to TextTwist.

**3**. The function $N$ takes a number and multiplies it by its conjugate. For example,

$$\begin{aligned} N(3+2i) &= (3+2i)(3-2i) = 13 \\ \text{and} \quad N(2-i) &= (2-i)(2+i) = 5. \end{aligned}$$

Find integers $a$ and $b$ so that $N(a+bi)$ matches each of the numbers below, or determine if it's impossible.

(a) 17
(b) 19
(c) 65
(d) 85
(e) 133

This function is sometimes called the "norm," but that term is not used consistently so we won't use it.

**4**. Put the following points in order of their distance from the origin, from closest to farthest.

(a) $O = (-5, -5)$
(b) $R = (0, -8)$
(c) $L = (6, -3)$
(d) $I = (7, 4)$

O RLI?

**5**. Put the following numbers in order of their $N$-value, from lowest to highest.

(a) $a = -5 - 5i$
(b) $r = 0 - 8i$
(c) $m = 6 - 3i$
(d) $k = 7 + 4i$

**6**. Tabulate the values of $(2+i)^n$ for $n = 1, 2, \ldots, 8$. Calculate the $N$-values for all the answers you obtained.

**7**. Find all Pythagorean triples whose hypotenuse length matches each of the numbers below.

(a) 5
(b) 25
(c) 125
(d) 625

### Neat Stuff

**8**. Find the two solutions to each of these equations.
(a) $x^2 - 16x + 63 = 0$
(b) $x^2 - 16x + 64 = 0$
(c) $x^2 - 16x + 65 = 0$
(d) $x^2 - 18x + 85 = 0$
(e) $x^2 - 16x + 145 = 0$
(f) $x^2 - 24x + 145 = 0$

**9**. For each pair of solutions that you found in problem 8, calculate their sum and product.

**10**. Tabulate the values of $(3+2i)^n$ for $n = 1, 2, 3, 4$. Calculate the $N$-values for all the answers you obtained. Try other numbers. It's fun!

**11**. Find a Pythagorean triple whose lengths have no common factors and whose corresponding hypotenuse length is $13^3$.

**12**. In class, we conjectured that any number that is one more than a multiple of 12 can be written as the sum of two squares (of integers). Does this always work?

**13**. Write each prime as $n = x^2 + y^2 - xy$, where $x$ and $y$ are integers, or determine that it's impossible.
(a) 101
(b) 127
(c) 419
(d) 421
(e) 10009

**14**. Write each number as $n = x^2 - 2y^2$, where $x$ and $y$ are integers, or determine that it's impossible.
(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

### Tough Stuff

**15**. Let $(x, y)$ be a point on the unit circle. If you walk along the circle from $(1, 0)$ to $(x, y)$, then walk that same distance farther along the circle, where will you be?

This question should use trigoNOmetry. As in, don't use that.

**16**. What prime numbers are squares in mod 17? (Include primes that are larger than 17.) What primes $p$ make 17 a perfect square in mod $p$?

**17**. Find all integer solutions to this system of equations.

There are probably more than you think.

$$\begin{aligned} a+b &= cd \\ c+d &= ab \end{aligned}$$

**18**. Prove that every positive integer not of the form $8n+7$ or $4n$ is a sum of three squares having no common factor.

Legend(re) proved this in 1798.

**19**. Time to get ridiculous.

(a) What fraction has decimal expansion 0.538461538461...?
(b) ... 0.461538461538...?
(c) ... 0.010203040506...?
(d) ... 0.020508111417...? (1 less than multiples of 3)
(e) ... 0.010102030508132134...? (Fibonacci)
(f) ... 0.01030927...? (Powers of 3)
(g) ... 0.0104091625...? (Square numbers)