2007.6 For the Love of the Game

Game of the Day: "Shell Game"

In the Shell Game, a prize ball is placed under one of four shells. The player is asked four true-false questions: for each one they get right, they get to place a marker next to any of the four shells. If they place a marker next to the prize ball, they win.

Once, the player accidentally looked under the shell instead of placing the marker. The prize ball wasn't there. They placed the marker there anyway . . . brilliant!

1. (a) Assuming that the true-false questions are all answered with 50-50 likelihood, what is the probability that the player answers all four questions right (and automatically wins)?

(b) Find the probability that the player gets 0 right, 1 right, 2 right, 3 right, 4 right.

(c) Use expected value to find the probability that the player wins the Shell Game. For example, if the player gets 3 true-false questions right, they will *then* have a $\frac{3}{4}$ chance of winning the game.

Mmm, symmetry.

Important Stuff.

2. If you have not yet done problem 7 from yesterday, do so now. Either way, use the calculator to find `nCr(4,k)` for each k from 0 to 4; also find the value of $\binom{10}{3}$ and $\binom{10}{7}$.

Shouldn't we have said "values" here?

3. There are 16 trains of length 5. Lars, apparently impersonating Butch Cassidy, steals one of the trains at random. What is the probability that the train he stole was made from exactly 3 rods?

Or David Cassidy, we forget which. Come on, get happy!

4. Flip four coins. What is the probability of getting exactly two heads?

5. Herb stands at the corner of Center St and Main in beautiful downtown Logan, Utah. Assume for the sake of this problem that Logan is a grid of streets (pretty close, actually). Center goes east-west, while Main goes north-south.

Herb flips a coin. If he flips heads, he goes east one full block on Center. If he flips tails, he goes west one full block. Then he flips again.

(a) After two flips, where could Herb be and with what probability?

(b) After three flips, where could Herb be and with what probability?
(c) After four flips? Five?
(d) Find the probability that Herb is back at the corner of Center and Main after ten flips. You'll probably want to use that `nCr` thing for this.

6. Quick calculator skill today: to get a fraction to appear in the entry window of the calculator, hit **ctrl** (the blue button) then the division key. Then you can type whatever you want in the numerator and denominator.

Use the calculators to expand this expression:

$$\left(x + \frac{1}{x}\right)^4$$

Any thoughts? What about the 10th power?

The other templates were found with **ctrl** then the multiplication key. Look back to Session 4 for instructions on how to expand stuff; but keep in mind you can type out the word `expand` like a function.

7. What's the probability that a randomly chosen positive integer is

(a) divisible by 3?
(b) divisible by 5?
(c) not divisible by 3?
(d) not divisible by 3 or 5?
(e) not divisible by either 3 or 5?
(f) divisible by either 3 or 5?

8. Pick two positive integers at random, and call them m and n.

(a) What is the probability that m is even? What is the probability that *both* m and n are even?
(b) What is the probability that m and n do *not* share a common factor of 2? You can use the result from part (a) to solve this.
(c) What is the probability that m and n do *not* share a common factor of 3?
(d) What is the probability that m and n do *not* share a common factor of 5?
(e) What is the probability that m and n do *not* share any of the common factors 2, 3, or 5?

What's the chance that they *do* share the factor?

Neat Stuff.

9. This table gives the number of elements for the Farey sequence of order n. The third column gives the difference between consecutive terms.

n	$\|F_n\|$	Δ
1	2	1
2	3	2
3	5	2
4	7	4
5	11	2
6	13	6
7	19	4
8	23	6
9	29	4
10	33	
11		
12		
13		
14		
15		

Continue this table to F_{15} by considering how many elements get added to the Farey sequence each time.

The notation $|F_n|$ just means the number of elements in F_n. As an example of how Δ works, look at 19, 23, and the 4. Either $23 - 19 = 4$, or $19 + 4 = 23$. When you go from F_7 to F_8, four terms get added for a new total of 23. Hm, this table must be useful if it's taking up half a page . . .

So, what would this look like as a graph?

10. So, there are 16 trains of length 5 and 16 ways to flip 4 coins. Can you think of a way to match them up? It would be something like "this coin flip corresponds to this train".

This is usually called a *one-to-one correspondence* and is a typical solution method in counting problems.

11. Using your correspondence from problem 10, what train is represented by this sequence of heads and tails?

HHTTTTHTHHHHH

Note there may be more than one correct answer here since it depends on the correspondence used.

12. In yesterday's Cross-Program Activity, an interview process was mentioned. Candidates come in, and you must decide to accept them immediately or reject them forever. The goal is to have the best chance of finding the one best candidate among the group. One strategy is to reject the first n candidates out of hand, then accept the next candidate that is better than anyone seen so far.

Another application of this process is finding a good parking space: you might pass up the first few spaces you find expecting to see a better one later, or take a spot only to find a closer one you could've had. Could also apply to relationships . . .

Suppose there are 5 candidates to be seen.

(a) Oscar decids to accept the first candidate. What is the probability that Oscar gets the best possible candidate?

(b) Judy decides to reject the first candidate, then take the next one *that beats everyone so far*. What is the probability that Judy gets the best candidate this way?

(c) Same question, but Allen throws out the first two candidates before looking for the best of the rest.

With technology you might consider expanding the problem to larger groups . . .

13. On average, how far is it from one prime number to the next? How would you go about measuring such a thing?

14. Today's Game of the Day is a little simplified, since the player usually does better than 50-50 on the true-false questions in Shell Game.

(a) Suppose the player gets a true-false question right with probability $p = 0.75$. What is the chance that they win the Shell Game?

(b) Find a formula, in terms of p, for the probability that the player wins the Shell Game if they get questions right with probability p. Check that the formula gives the right answers for $p = 0, p = 0.5, p = 1$.

Tough Stuff.

15. Take the harmonic series and remove all the terms with the number 1 in their denominators:

Sadly this means the fraction $\frac{1}{231}$ is missing.

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{9} + \frac{1}{20} + \frac{1}{22} + \frac{1}{23} + \cdots$$

The normal harmonic series diverges (gets larger with no maximum). So, what about this one?

16. Take the harmonic series but consider only terms with prime denominators:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

Does this sum converge? If so, to what? If not, can you prove it?