2007.11 Oh, The Games You'll Play!

Game of the Day: "Wheel of Fish"

On the classic show "Wheel of Fish", players spin a wheel and earn a number of fish based on their spin. The four stops on the wheel are

one fish

two fish

three red fish

ten blue fish

They can then exchange their fish for the contents of a mystery box, but that's really beside the point. We realize you know how to play, but we're going to invite contestants up here just for the halibut.

1. Find the mean, variance, and standard deviation for the number of fish earned from one spin of this wheel.

2. Find the mean, variance, and standard deviation for the number of fish earned from two spins of this wheel, with the 16 possible outcomes

$$2, 3, 4, 11, 3, 4, 5, 12, 4, 5, 6, 13, 11, 12, 13, 20$$

Avery points out that 2^4 and 4^2 are the same. Wow! Impressive. If only it worked for 5.

3. Expand this on the nSpire:

$$(f + f^2 + f^3 + f^{10})^3$$

Find the probability that you earn exactly 6 fish in three spins.

Important Stuff.

4. (a) How many numbers less than or equal to 15 do *not* share a common factor with 15?

(b) How many numbers less than or equal to 35 do *not* share a common factor with 35?

(c) How many numbers less than or equal to 91 do *not* share a common factor with 91?

(d) Multiply this out:

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

In our "Dead Giveaway" department, the numbers for 4(a) are: 1, 2, 4, 7, 8, 11, 13, 14. Oh and by "common factor" we mean "not 1".

5. Consider the numbers 1 through 35.

(a) What fraction of these numbers are divisible by 5? by 7?

(b) What fraction of these numbers are *not* divisible by 5? by 7?

(c) Write down the numbers 1 to 35. Cross out any number that is divisible by 5. What fraction of the original 35 numbers remain?

(d) Now cross out any number that is divisible by 7. What fraction of the numbers that survived part (c) also survived this second cut?

(e) What fraction of the original 35 numbers survived both cuts?

Usually people stop at 10 or 20 or 100, so the numbers 1 through 35 rarely get proper consideration. Not much to say about 35 really. It's one more than that miracle street, and one less than a square triangular number. Poor guy. At least 35 is the highest number you can reach counting on your fingers in base 6.

6. So there's this function ϕ that takes positive integers as input. You take the input and find all its prime factors. For each prime factor, you multiply through by $(1 - \frac{1}{p})$. For example, take 15. The primes are 3 and 5, so the result is

$$\phi(15) = 15\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

(a) Calculate $\phi(15), \phi(35)$, and $\phi(91)$.

(b) Calculate $\phi(105)$ and $\phi(231)$.

(c) Calculate $\phi(9)$ and $\phi(27)$. Watch out!

(d) Calculate $\phi(8675309)$.

(e) Describe what ϕ measures in your own words.

Naturally it would. This function is pronounced "fee" of n, which you might hear if a giant approaches an Englishman.

The nSpire can factor numbers if you need it to. It also makes Julienne fries.

7. Complete this table for $\phi(n)$ along with the cumulative total of all ϕ values from 1 to n.

n	$\phi(n)$	$\sum \phi(n)$
1	1	1
2	1	2
3	2	4
4		
5		
6		12
7		
8		
9		
10		32

How many elements are in F_8, the Farey sequence of order 8? What about F_9? Hm.

8. Describe and explain some connections between the ϕ function and the Farey sequences.

Neat Stuff.

9. Look back at the lists of Farey sequences from earlier in the course.

Kids like adding fractions this way. But then again, they also like paste and Pokemon.

(a) What is the first fraction placed between $\frac{2}{5}$ and $\frac{1}{2}$?
(b) What is the first fraction placed between $\frac{2}{7}$ and $\frac{1}{3}$?
(c) What is the first fraction placed between $\frac{2}{5}$ and $\frac{3}{7}$?
(d) What is the first fraction placed between $\frac{3}{5}$ and $\frac{5}{8}$?
(e) Prove that if $\frac{a}{c} < \frac{b}{d}$ for positive a, b, c, d, then

$$\frac{a}{c} < \frac{a+b}{c+d} < \frac{b}{d}$$

10. Suppose you were standing at the origin $(0, 0)$ and there were 900 points of light, one at every point (x, y) with integer coordinates 30 or less. You pan across from the east to the north, and in between you see a whole lot of points. You can't see any point blocked by another one: for example, you can't see $(24, 21)$ because $(8, 7)$ is in the way.

We refuse to do a Madonna joke.

(a) Approximately what percentage of all 900 points do you see?
(b) What are the very first points you see?
(c) What is the first point you see that *doesn't* have y-coordinate 1?
(d) What point do you see exactly halfway along this panning? By this we mean in terms of points seen, not in terms of angle.
(e) What point is halfway through the "first half" of the panning? Can you explain why? No trig is needed.

At this point, Sendhil might sing, "Whoa, we're halfway there . . . "

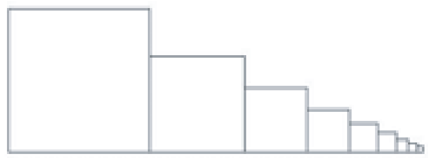
At this point, Sendhil might sing, "Whoa, we're approximately one-fourth of the way there . . . "

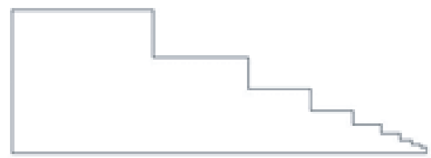
11. (a) You spin the Wheel of Fish four times. What's the most likely number of fish you'll win? Note that this is not the same as the mean, or even the median.
(b) What's the most likely number of fish from *ten* spins of the wheel?

Guh! What a mess! And I'm just talking about the big pile of fish.

11. Cathy imagines an "infinite stairway", created by lining up an infinite number of squares. The first square is 1 by 1, the second is $\frac{1}{2}$ by $\frac{1}{2}$, the third is $\frac{1}{3}$ by $\frac{1}{3}$, and the nth is $\frac{1}{n}$ by $\frac{1}{n}$

And she's buying a stairway to 11. Or, maybe this would be better as a Spinal Tap reference?





So, what's the perimeter and area of this infinite staircase?

Tough Stuff.

12. Build a histogram for the number of ways (or the probability, take your pick) you can get each possible outcome from 10 spins of the Wheel of Fish. For example, there are 49,905 ways to spin the wheel 10 times and earn exactly 36 fish.

A bar graph is *unacceptable*!

13. Can $\phi(n)$ ever be less than $\frac{n}{10}$? Explain.

14. So now you know (perhaps from the calculator) that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

What about the sum of reciprocals of *odd* squares only? That is,

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$$

or in summation form

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Try figuring this out without the use of technology.

15. So $2^4 = 4^2$, and that's interesting: it's the only pair of positive integers x and y with $x^y = y^x$ and $x \neq y$. When else does $x^y = y^x$ when $x \neq y$ for positive reals x and y? Never? Ever after? Graph?

What a cool slash for not equal. Is that the same slash as the one in GnR?