7 *Exit the Matrix...*

This problem set was compiled on a spiffy new Lappy 486 computer! The Lappy includes a whopping one-half of ten minutes battery life.

PROBLEM

Start with a 30-60-90 triangle with hypotenuse 12. Attach another 30-60-90 triangle so that its hypotenuse lines up with the longer leg of the original (the new triangle is a little smaller).

Now do this, over and over again... over and over again...

1. What is the area of the sixth triangle?
2. What is the sum of the areas of all six triangles?
3. If you kept doing this forever, what would the sum of *all* the triangles' areas be? Include overlaps (double, triple, etc. counting).

Useful Stuff.

1. Where we're going, we won't need skis (but you will need to look at the map on the board).
   (a) According to the map, how many ways are there to get from Altavista to Payday in two steps?
   (b) How many ways are there to get from Altavista to Payday in exactly three steps?
   (c) How many ways are there to get from the Lodge to Ginny's Slope in exactly two steps?

Note: stumbling down the side of the mountain on a bike does *not* count as a step.

(d) How many ways are there to get from Canyon Pass to Eagle in exactly three steps?
(e) How many ways are there to get from Altavista to the Lodge in *no more than* three steps?

2. Build a six-by-six table of numbers that you can use to convey all the information from the map. This time, we're going to ask you to be consistent: each row across should say how many ways there are to go *from* X *to* Y in a single step.

Use any order you like for the six locations, but we will present them using this order: Payday, Canyon Pass, Altavista, Ginny's Slope, Lodge, Eagle.

3. Use the table, and only the table, to build another six-by-six table for the two-step paths *from* X *to* Y. Be sure you can describe this entire process; saying "I squared this" is *not* what we're looking for.

Unfortunately, no one can be *told* what a matrix is. You have to experience it for yourself.

4. Use the table of one-step paths *and* the table of two-step paths to build a table of three-step paths.

5. Learn how to Enter The Matrix on a calculator, then tell the calculator to square a matrix. Hey, check it out.

6. Using a calculator, determine the number of paths that start and end at the Lodge that take *no more than* 5 steps. It's fine to just describe *how* to do it if you're not interested in actually performing the calculation.

Choose between a TI-83, TI-83+, TI-84, TI-84 Platinum, TI-89, TI-89 Titanium, TI-92, TI-92+, or a Voyage 200. Or better yet, just use MATLAB. Teaching note: most TI calculators have a 9-by-9 limit on matrix size.

Neat Stuff.

7. There are only two roads in Strong Badia: a one-way looping road that goes around the town, and a two-way road to and from the airport.
As a taxi driver, Homsar needs to start and end in downtown, but he knows he'll travel on roads 9 times. How many options does he have?

So, one option is "Travel to the airport, then back, then do the downtown loop 4 times, then to the airport, then back, then the downtown loop one more time." Homsar loves Sammy Hagar tunes, by the way.

8. A proposed change to the Strong Badia roads would add four new one-way downtown loop options (giving a total of five) and thirteen new one-way roads to the airport (giving a total of 14 ways to go from downtown to the airport). There will still only be the single road from the airport to downtown.
If this change goes through, how many options will Homsar now have if he'll travel on 2 roads? 3 roads? 4 roads? 5 roads? Look familiar? What's the deal?!

Exit the Matrix. . .

9. Say, what's the result of this matrix calculation:

$$\begin{bmatrix} 0 & 1 \\ 14 & 5 \end{bmatrix}^3$$

265, eh? I think we saw that number recently.

10. **Luddite ALERT!** *Without* using any technology more advanced than the 20th-century ballpoint pen, calculate this:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^9$$

Neat Stuff.

11. You invest \$1,000 at 5% APR.
(a) Use the iteration rule $x \mapsto 1.05x$ to determine how much money you'll have at the end of 10 years.
(b) Give a closed-form rule that says how much money you'll have at the end of n years.

12. You invest \$1,000 *each year* at 5% APR, and will cash it all out after 10 years.
(a) How much will the money you put in today be worth in ten years? Give an "exact" answer, not some decimal.
(b) Repeat the question for the money you invest one year from now. How much will it be worth at the end of the ten years?

How long is this money getting invested anyway?

(c) Write a nice long expression for the total money you'll have at the end of the ten years.
(d) Any thoughts on simplifying this thing? cough cough geometric sequence cough

13. Consider the iteration rule $x \mapsto -0.75x + 12$.
(a) Pick a number and iterate. What could happen?
(b) What are the fixed points, if any?

14. Consider the iteration rule $x \mapsto -1.1x + 3$.
(a) Pick a number and iterate. What could happen?
(b) What are the fixed points, if any?

15. Consider the iteration rule $x \mapsto -x + 12$.
(a) Pick a number and iterate. What could happen?
(b) What are the fixed points, if any?

16. Consider the iteration rule $x \mapsto 1.01x - 3$.
(a) Pick a number and iterate. What could happen?
(b) What are the fixed points, if any?

Bored with these questions yet? Too bad!

17. Consider the iteration rule $x \mapsto Ax + B$.
(a) Pick a number and iterate. What could happen? Note: there are a lot of answers here, depending on A and B — we want them *all*.
(b) What are the fixed points, if any? Can there ever be more than one fixed point? Less than one?

Tough Stuff.

18. Homsar drives randomly about the town (now that all these extra roads have been built). As a very long day wears on, what's the probability he'll be on the way to the airport at any given time? Assume all the roads take the same amount of time to travel upon.

19. A bunch of skiers get on at Lodge, and they choose paths at random (equally likely to take any path). As the day continues, skiers start to fan out over the 6 mountain locations (assume no time between locations, so everyone is at one of the 6 locations). What's the probability of finding a particular skier at Ginny's Slope after, oh, say, 100 steps? (The number of steps turns out to be unimportant as long as it's big enough.)

20. Consider the iteration rule $x \mapsto x^2 + B$.
(a) For what numbers B will there be fixed points? (Here, x is restricted to real numbers.)
(b) For what numbers B will there be *attracting* fixed points?
(c) Take the starting number $x = 0$. As B changes, the behavior that $x = 0$ follows in the iteration changes. Describe the changes in as much detail as possible, and find the smallest number B so that $x \mapsto x^2 + B$ won't eventually dive off to infinity when starting with $x = 0$.

For example, $B = -10$ fails, since the iteration goes $0 \mapsto -10 \mapsto 90 \mapsto 8090 \mapsto$ BIG...

21. Write your own problem that links matrices, maps, and genomics. Or, steal a good one from some other book that we can use in the next couple of days.