12 *Fun with Matrices*

PROBLEM

Tristen buys a car. The car costs \$12,000, and the financing is 6% APR, divided into 0.5% every month. A car payment is due every month for 36 months.

(a) Suppose Tristen paid \$300 per month. Show that this isn't enough to pay off the car. How much would be left?

(b) Suppose Tristen paid \$400 per month. Show that this is more than enough to pay off the car. How much extra was paid?

(c) Find the correct car payment, to the nearest penny.

We were going to write this problem about Kelley, but. . . nah. We also wanted this to be an annuity problem, but we didn't want anyone to lose interest.

Useful Stuff.

1. A square lies with the corners $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Say, we could represent those points as the columns in a 2-by-4 matrix.

(a) Type that matrix into a calculator, but only if you feel you could do everything without it. Call this matrix George. No, wait, call it A.

(b) Compute $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot A$. Draw the new shape and find its area.

(c) Compute $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot A$. Draw the new shape and find its area.

Hacksaw Jim Duggan was once known for beating people over the head with a 2-by-4 matrix. So were the Three Stooges.

(d) Compute $\begin{bmatrix} 5 & 2 \\ 3 & 7 \end{bmatrix} \cdot A$. Same same.

(e) *A little tougher:* Find the area of the shape that results when you multiply by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. You can assume all of a through d are positive. Hurray for parallelograms.

So about this area, can you determine it?

2. Take the matrix $B = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix}$. Here's a set of starting points P. For each P, calculate $B \cdot P$, then $B^2 \cdot P$, then $B^3 \cdot P$. Dude, read the side note.

Important! You may use a calculator here, but *not* for matrices. So, feel free to use it as a crutch for adding and multiplying, but there's something important in this problem you might not catch if you just slam slam slam buttons. After all, we want you to do things... over and over again...

(a) $P = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

(b) $P = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(c) $P = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(d) $P = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(e) Hey, you. Look back. What's the deal here?

(f) Answer the questions for $P = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$ without doing any matrix calculations.

3. Take the matrix $C = \begin{bmatrix} 0 & 1 \\ -35 & 12 \end{bmatrix}$. Consider the starting point $P = \begin{bmatrix} 1 \\ r \end{bmatrix}$.

(a) Write $C \cdot P$ in terms of r.

(b) What two values of r will make $C \cdot P$ a scaled copy of P? In other words, what makes $C \cdot P$ a multiple of P?

Lynda points out that if this was the last problem, the answers would've been $r = 5$ and $r = 2$.

4. Describe what the matrix

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

does to any point in the plane.

5. The matrix

$$D = \begin{bmatrix} 0 & 1 \\ 8 & 2 \end{bmatrix}$$

can also scale points. Find some points, other than $(0, 0)$, that scale when multiplied by this matrix. In other words, solve this system:

$$\begin{bmatrix} 0 & 1 \\ 8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

6. Pick any starting point and run it through matrix D of problem 5. What happens after taking a high power of D? Any relationship to the old recursive stuff we did?

Neat Stuff.

7. Lars can afford \$450 per month for a car, now that he rules the Park City math world (49 weeks per year).

For 3 weeks, he has to compete with Herb. . . and that's a tough one.

(a) A 3-year loan can be had at 3.9% APR. What's the price of the most expensive car he could get with \$450 per month?

(b) A 4-year loan can be had at 4.9% APR. What's the price of the most expensive car he could get with \$450 per month?

(c) A 5-year loan can be had at 5.9% APR. What's the price of the most expensive car he could get with \$450 per month?

8. Find the two numbers k that can make this equation true:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

Whoa, what's up with that?

9. Look back at the ski map from Day 6. Back then, we built a matrix saying how many *ways* there were to go from X to Y for each location.

There's another kind of matrix that can be built with the *probability* of going from X to Y at any given time. So, say you randomly decide where to go once you get somewhere. Build a 6-by-6 matrix with the *probability* of going from X to Y, given that you are already at X.

This means that each row's values should add up to 1, since it's the total probability that you go from X to anywhere.

10. Alright, now take the fourth power of the matrix you got in problem 9. Interpret the meaning of this matrix. Does it matter where you started?

11. Alright, now take the 50th power of the matrix from problem 9. What is happening?

Warning! Slow calculators stink!

12. Build a probability matrix for the map from the Good Will Hunting problem (day 10). Take that matrix to the 50th power. Does the same thing happen?

13. Hey, experiment with this iteration rule, where $0 \le x \le 1$:

$$x \mapsto 2.5x(1-x)$$

14. Same thing, except $x \mapsto 3.1x(1-x)$. What's up with that?

15. Algebraically, find the fixed points of this iteration:

$$x \mapsto kx(1-x)$$

Criminy, when will we stop asking "What's up with that?" What *is* up with that?

Tough Stuff.

16. Hey, take a cube with one vertex at $(0,0,0)$, side lengths 1, and all volume in the first "octant". So, then you multiply by a 3-by-3 matrix, and the volume changes. Describe how. Bonus prizes if you build the 3-D parallelogram equivalent, the "parallelpiped" if you will. Big bonus if you use Zomes.

Basically, the square from the first problem in fabulous 3-D.

17. Find the closed-form rule for

$$\begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix}^n \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

in terms of x, y, and n.

18. Use a system of equations to find the exact probability of being at the six different ski locations *in the long run*. Oh yeah, same for Good Will.

21. Experiment with this recursion:

$$t_n = 2x \cdot t_{n-1} - t_{n-2}$$

where the starting point is $t_0 = 1$, $t_1 = x$. Generalize the zeroes of Pafnuty's n^{th} polynomial.