Circle Packing: A Directed Investigation of Descartes' Circle Theorem

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What Is It?

Descartes' Circle Theorem involves relationships among radii of tangent circles, "packed" together. The relationship is expressed using the curvature of each circle, which is also the reciprocal of the radius involved. This article contains background and material with problems related to Descartes' Circle Theorem.

Grade Level/Strand:

The lesson is designed for high school geometry students who have studied tangent circles.

Class Time:

The lesson is designed to be done outside of class as a directed investigation.

Materials:

No materials are required but the investigation can be done with the help of *The Geometer's Sketchpad*.

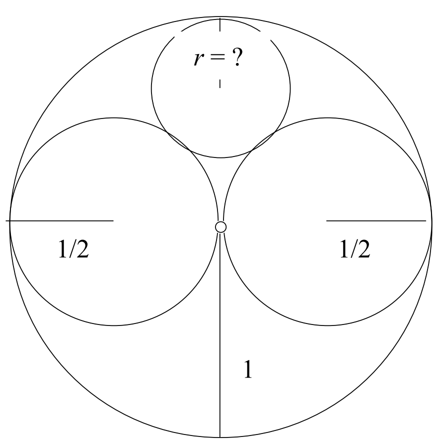
Prerequisites:

Students should be facile with *Geometer's Sketchpad* and should be prepared to solve quadratic equations.

Introduction:

Mathematics is not so established that there is nothing new to be learned, especially from high school geometry. While at a conference in 1998, Allan Wilks, a statistician at AT&T Laboratories in New Jersey, had a conversation with a colleague who had been pondering his daughter's homework assignment. The question concerned a pattern made up of a large circle (of radius 1) tangent to two congruent, "kissing" (tangent) circles (of radius _) that fit inside it. The problem was to find the radius (r) of the smallest circle tangent to all three existing circles as in Figure 1.

Figure 1. Nested tangent circles



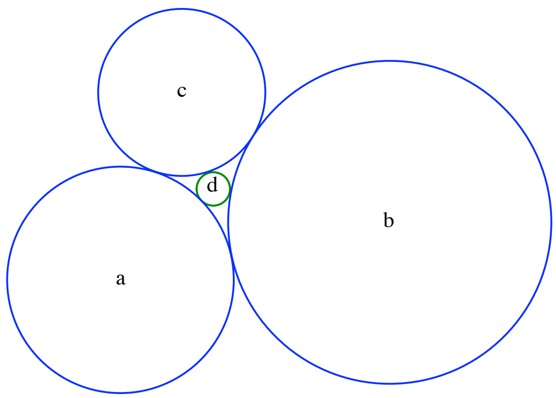
Wilks contemplated the circle problem after the conference ended. He was curious about the relative sizes of the touching circles. And he was not the first mathematician to become engaged in the problem.

In 1643, French mathematician Rene Descartes developed a formula relating the curvatures of four tangent circles. (Coxeter, 1969)

Descartes' Circle Theorem

Given four mutually tangent circles with curvatures a, b, c, and d as in Figure 2, the Descartes Circle Equation specifies that $(a^2 + b^2 + c^2 + d^2) = (1/2)(a + b + c + d)^2$, where the *curvature* of a circle is defined as the reciprocal of its radius.

Figure 2. Mutually tangent circles

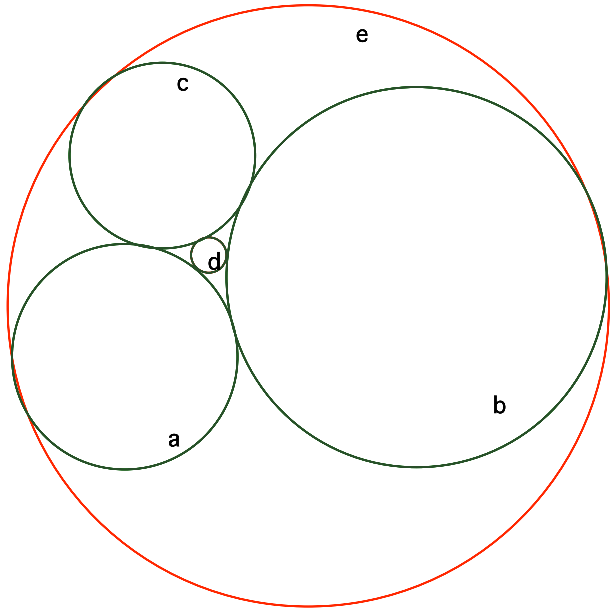


Another way of looking at curvature of a circle is by forming a ratio comparing the circumference of the unit circle to the circumference of the given circle. If the radius of a circle is 3, then the curvature would be $2\pi/6\pi$ or 1/3.

Curvature, $1/r$, for a circle of radius r also describes how fast you are turning as you walk any part of the circumference of that circle. If you walked a certain distance d on a circle then you would turn by angle $\theta = d(1/r)$, since
$d = r\theta$.

Descartes' Circle Equation can be used to specify the size of every subsequent, smaller circle that fits into the tangent circle pattern. For example in Figure 3, if the curvatures a, b, and c are known, then this equation is a quadratic equation in the unknown curvature with one root equal to the curvature, d.

Figure 3. Two Sets of Mutually Tangent Circles



As in both Figures 1 and 3 with one circle enclosing other circles, an adaptation of the definition of curvature has to be adopted for the Circle Equation to hold. That is, the quadratic equation holds if the curvature of the enclosing circle is negative (curvature of the larger circle = –1 in Figures 1 and 3). In lay terms, if all the points of tangency are external, the curvatures are considered positive, but if one circle encompasses the others, that circle has negative curvature. In Figure 3, curvatures *a, b*, *c*, and *d* have positive curvature while curvature *e* is negative. Applying Descartes' Circle Theorem in Figure 3 with curvatures, *a*, *b*, *c*, *d*, and *e*, the curvatures *d* and *e* are the positive and negative roots of the following quadratic polynomial in *x*:

$$(a^2 + b^2 + c^2 + x^2) = (1/2)(a + b + c + x)^2.$$

To show this, consider that Descartes' Theorem applies to both circles with curvatures *d* and *e* since both are tangent to *a*, *b*, *c*. There is nothing in the theorem about being tangent inside or outside, except for the sign convention.

Wilks realized because the Root Sum Theorem for quadratics states that the roots *r* and *s* of polynomial $ax^2 + bx + c = 0$ have product *c* and sum –*b*, and because if *b* is an integer and one of the roots is an integer, then the other root must also be an integer. Thus, if the curvatures of the four initial circles are integers, the other root *e* is an integer. Furthermore, if additional tangent circles are constructed to fill in the holes in such a figure, the curvature of every subsequent tangent circle is also an integer.

Other people have been fascinated by and used the Descartes Circle Equation. Nobel Prize winning Chemist Frederick Soddy (1936) wrote "The Kiss Precise" about Descartes' Circle Equation with a middle verse as follows:

Four circles to the kissing come,
The Smaller are the benter,
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb

There's now no need for rule of thumb,
Since zero bend's a dead straight line
The sum of the squares of all four bends
Is half the square of their sums.

A connection to Wilks' contemplation and Descartes' Circle Equation arose at the Park City Mathematics Institute 2002. Jeff Lagarias, a mathematical researcher at AT&T, introduced me to the topic of packing circles in an informal lunch conversation when I told him of my fascination with the morning theoretical mathematics sessions using Gaussian integers (complex numbers in the form $a + bi$ where a and b are integers). Jeff showed me the original circle-packing problem but went a step farther to show me a connection to Gaussian integers. He related that a Gaussian integer is produced for every circle when its center (in a complex plane) and its curvature are multiplied. While this Gaussian integer connection is intriguing, most high school students are not ready for that connection. However, to get them to begin thinking about tangent circles, the following worksheets focus only on curvature and Descartes' Circle Equation while their teachers are encouraged to explore the Gaussian connection (see the Wilks reference below).

References:

Coxeter, H. M. S. 1969. *Introduction to Geometry*. New York: John Wiley & Sons, Inc., pp. 11-13.

Further information on Wilks' findings, curvature and Gaussian integers can be found at http://www.sciencenews.org/20010421/bob18.asp

Peterson, I. 2001. Circle Game. *Science News Online* (April 21). Available at http://www.sciencenews.org/20010421/bob18.asp

Soddy, F. 1936. "The Kiss Precise," *Nature* (137), 1021.

Sykes, M. 1912. Sourcebook of Problems for Geometry. Palo Alto, CA: Dale Seymour Publications.

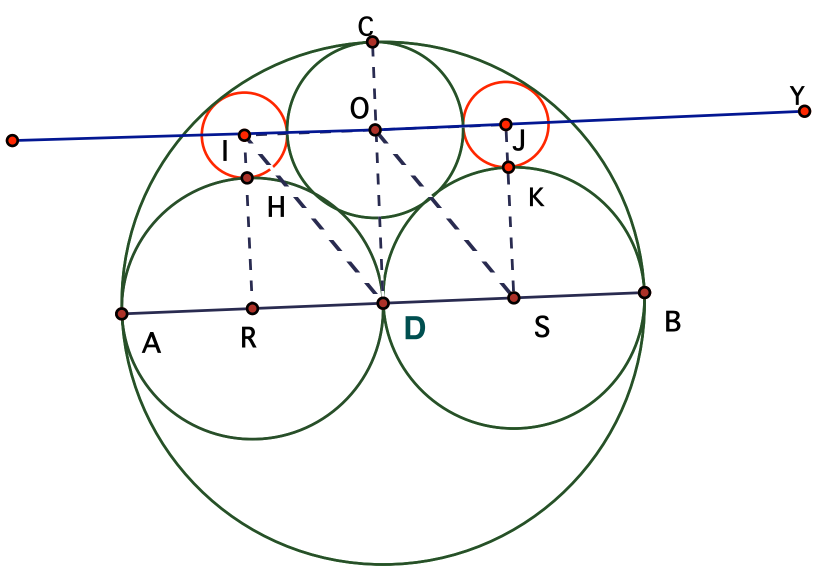
ANSWERS TO QUESTIONS

1. 6
2. 2
3. 5
4. 3
5. Smaller
6. 1
7. _; 2
8. _; 2
9. 1/3; 3
10. 2
11. 2
12. 15
13. 3; 1/3

14. 2
15. 2
16. 3
17. 1; –1
18. No.
19. 1/6
20. 6
21. 1/6
22. 6
23. A rectangle.
24. Perpendicular bisector of segment *AB*.
25. Construction:

Given segment *AB*, construct the figure.

1. Draw segment *AB*.
2. Construct midpoint of segment *AB* at *D*.
3. Using *D* as center and *DA* as radius, draw circle *D*.
4. Place point *C* where perpendicular bisector of segment *AB* and circle *D* meet.
5. Construct midpoint of segment *AD* at *R* and midpoint of segment *DB* at *S*.
6. Construct circles: circle *R* with radius *RD* and circle *S* with radius *SD*.
7. Trisect segment *CD*, so that 2CO = OD.
8. Through *O*, construct segment *OY* parallel to segment *RD*.
9. Label the intersection of segment *OY* and the perpendicular to segment *AB* at *R* as point *I*. Label the intersection of segment *OY* and the perpendicular to segment at point *S* as point *J*. Let *H* be the point of intersection of segment *RI* and circle *R*; let *K* be the point of intersection of segment *JS* and circle *S*.
10. Draw circle *I* with radius *IH* and circle *J* with radius *JK*. Label the point of tangency of circles: *I* and *R* point *H*. Label the point of tangency of circles *J* and *S*: point *K*.



* This construction could be simplified to involve only circles *D*, *R*, *S* and *O*.

** Another way to devise the construction of this figure could be by working backwards from rectangle *RIJS*. This is easiest if the sides have lengths 1 and 2/3 with a ratio of 3:2. The circles would then be constructed using the vertices of the rectangle and the radii deduced in previous problems.

***An interesting addition might be to ask the students how they could prove that circle *J* and circle *S* are actually tangent. (*JK* + *KS* = *JS*)

26. *D*: $r = 1/102$
    $c = 102$
  E: $r = 1/6$
    $c = -6$
  F: $r = 1/15$
    $c = 15$
  G: $r = 1/26$
    $c = 26$
  H: $r = 1/110$
    $c = 110$
  I: $r = 1/51$
    $c = 51$
  J: $r = 1/42$
    $c = 42$
  K: $r = 1/86$
    $c = 86$

The following is the solution for the curvature of the first tangent circle given the 3 circle values in the original problem.

$$a^2 + b^2 + c^2 + d^2 = (1/2)(a + b + c + d)^2$$
$$14^2 + 11^2 + 23^2 + x^2 = (1/2)(14 + 11 + 23 + x\_^2$$
$$846 + x^2 = (1/2)(48 + x)^2$$
$$846 + x^2 = (1/2)(2304 + 96x + x^2)$$
$$1692 + 2x^2 = 2304 + 96x + x^2$$
$$x^2 - 96x - 612 = 0$$
$$(x - 102)(x + 6) = 0$$
$$x = 102$$

** It is also possible to employ the Root Sum Theorem to determine the solution.
If $x^2 + bx + c = 0$ and $(x - r_1)(x - r_2) = x^2 - x(r_1 + r_2) + r_1 r_2$, then $-b = r_1 + r_2$.

Answer to Extended Question: 1/6
Area of semicircle *ACB* is $\pi/2$.

Areas of circles *R* and *S* are $\pi/4$ (only need half of these circle areas because of location in semicircle).
Area of circle *O* is $\pi/9$.

Area of circles *I* and *J* are $\pi/36$.

Combined area of circles and semicircles in semicircle *ACB* is $5\pi/12$.

$$2(1/2)(\pi/4) + \pi/9 + 2(\pi/36) = \pi/4 + \pi/9 + \pi/18 = 5\pi/12$$

Area not occupied in semicircle *ACB* = $\pi/12$

Area of semicircle *ACB* – Combined area of circles and semicircles in *ACB*

$\pi/2 - 5\pi/12 = \pi/12$, which is 1/6 the area of semicircle *ACB*.